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A Method for Approximation and Control of Non-linear Systems

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Abstract. The article is concerned with the control problem synthesis for non-linear systems using linear techniques. The proposed approach consists on an approximation of a non-linear system by a set of uncertain linear systems. Control laws of these models are designed using classical methods. The first step in this approach is given by approximation of the non-linear system into a set of uncertain linear systems. The second step is the use of robust control methods based on LMIs that assure local stability in a non-infinitesimal state space domain. This last point permits to determinate the global control law. The global control law is based on gain scheduling form the measured state of the system. This approach is illustrated by a numerical example.

Keywords. Non-linear systems, HL-CPWL, Robust Control, Uncertain system, Norm bounded uncertainties, Gain scheduling.

1. Introduction

The use of non-linear systems in control has a long history especially in industries. This type of control has become an important domain of research which attracts many researchers. The application of this approach poses two problems :

- How to transform the non-linear system with a whole of linear models.
- How to guarantee the closed-loop global stability of the switched multiple linear system.

In recent years, several techniques of non-linear approximation systems have appeared; for example the high level canonical piecewise linear representation [8] [9]. The models obtained after this approximation are linear models. But these linear models do not make it possible to assure the transition of stability between two operational points. In this article, a solution for such problem is provided. In fact, we propose a new approach which transforms the certain linear models into a uncertain

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linear models and assures at same time the global stability of this non-linear system. The main idea of the new transformation is to add to each linear model an elliptic uncertain domain defined by a norm bounded uncertainty.
Then, we determine a control law of state feedback which guarantees stability in closed loop and ensures the transition between two points from operation.

This article is organized as follows: Section II will point out the approximation of non-linear systems. Section III will describe a non-linear control using the multi-model approach. Finally in section IV, a numerical example of control the physical system will be developed in order to validate an approach which guarantee the stability.

2. Approximation of non-linear systems

In this section, we give the definition of basic tools necessary to present the approximation of nonlinear systems by a family of the linear models. The high level canonical piecewise linear representation are the foundation of this multi-model approach.

2.1. Approximation of non-linearity

In this subsection, the construction of the basis for PWL functions proposed in [8] is briefly recalled. The domain considered in [9] is a rectangular compact set (1)

$$S = \left\{ x \in IR^n : 0 \le x_i \le m_i \sigma_i , \quad i = 1, 2, \ldots, n \right\} \tag{1}$$

where σ_i is the grid size and m_i which is subdivided using a simpliciale boundary configuration H (see [8] and [9]).

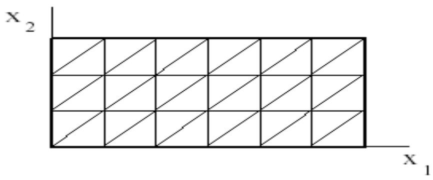

Fig1 : A simplicial Partition of a domain in IR^2

The building block of the basis is a generating function as follows:

$$\gamma(x_1, x_2) = \left(\left| \left| -x_1 \right| + x_2 \right| \right) - \left(\left| -x_1 + \left| -x_2 \right| \right| \right) + \left| -x_1 \right| + \left| x_2 \right| - \left| -x_1 + x_2 \right| . \tag{2}$$

Using (2), the functions $\gamma^0(x_1) = x_1$, $\gamma^1(x_1) = \gamma(x_1, x_1)$, $\gamma^2(x_1, x_2) = \gamma(x_1, x_2)$.
Generally is introduced

$$\gamma^k(x_1, x_2, ..., x_k) = \gamma(x_1, \gamma^{k-1}(x_2, .., x_k)) \tag{3}$$

A distinctive property of (3) is composed by k nestings of absolute value functions and accordingly it is said to have an n.l. equal to k .

The first element of the basis is a constant term $\gamma^0(1)$, and the others follow from a composition of $\gamma^k(.,...,.)$, $k = 1, 2, ...n$, with the linear functions :

$$\pi_{k,j_k}(x) = x_k - j_k\delta,\ k = 1, 2, ..n,\ j_k = 0, 1, ..m_k - 1\ .$$

As a result, the basis can be expressed in a vector form, ordered according to the n.l., as

$$\Lambda = [\Lambda_{0,s}^T, \Lambda_{1,s}^T, ..., \Lambda_{n,s}^T]^T$$

where Λ_i is a vector containing the $n.l. = i$ functions.

<u>Definition [9]</u>**:** The HL-CPWL are defined as follows, $f \in PWL_H[S]$ can be written as

$$f(x) = C^T \Lambda(x) \tag{4}$$

Where $C = [C_0^T, C_1^T, ..., C_n^T]^T$ and each vector C_i is a parameter vector associated with the $n.l. = i$ vector function Λ^i [when necessary, the jth component of vector $\Lambda(x)$ will be simply referred to as $\gamma_j(x)$].

After these considerations we define $f \in PWL_H[S]$, the approximation of a non-linear function g and it satisfies the following condition:

$$f(v_i) = g(v_i), \quad \forall v_i \in V_S$$

Let $V = \{v_1, v_2 ......, v_m\}$ be a set of m vectors $v_i \in \Re^n$, for $i = 1, 2, ...m$.

$$f(x) = C^T \Lambda(x) = \begin{bmatrix} f(V_S^0) \\ f(V_S^1) \\ . \\ . \\ . \\ f(V_S^n) \end{bmatrix} = \begin{bmatrix} \Lambda^{0^T}(V_S^0) & . & . & . & \Lambda^{n^T}(V_S^0) \\ . & & . & & . \\ . & & & & . \\ . & & & . & . \\ \Lambda^{0^T}(V_S^n) & . & . & . & \Lambda^{n^T}(V_S^n) \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ . \\ . \\ . \\ C_n \end{bmatrix} \tag{5}$$

The approximation of a non-linear function $g(x)$ have the following form $C = A^{-1}B$,with $B = [f(V_S^0), ..., f(V_S^n)]^T$, $C = [C_0^T, C_1^T, ..., C_n^T]^T$ and A an invertible and triangular square matrix.

Let's note that each nonlinear function with N-argument will be approximated by linear equations having the following form.

$$f(x)=\sum_{k=1}^{N}[(a_{1,k}x_1+b_1)+(a_{2,k}x_2+b_2)+....+(a_{n,k}x_n+b_n)] \quad (6)$$

with n *and* N represent respectively the argument number of the non-linearity and the number of linear model.

2.2 Determination of certain multi-model

The state representation of each non-linear system is represented in the following form:

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{cases} \quad (7)$$

where $u(t)\in\Re^{m}$ is the control vector. $A\in\Re^{n\times n}$ the dynamic matrix and $B\in\Re^{n\times m}$ the control matrix.

One will suppose for the continuation that the control matrix B is a constant matrix and that the non-linear function belongs to dynamic matrix A. The non-linear function “ g ” is approximated by a whole of the linear functions.
The state representation of the whole linear systems is as follows:

$$\begin{cases}\dot{x}(t)=\begin{pmatrix}\partial_{11} & . & . & \partial_{1n}\\ . & . & & \\ . & & . & \\ \partial_{n1} & & & g\end{pmatrix}x(t)+Bu(t)\\ y(t)=Cx(t)\end{cases}=\begin{cases}\dot{x}(t)=\sum_{i=1}^{N}\left[\begin{pmatrix}\partial_{1,1} & . & . & \partial_{1,n}\\ . & . & & \\ . & & . & \\ a_{1,i} & & & a_{n,i}\end{pmatrix}x(t)+\begin{pmatrix}0\\ .\\ .\\ b_1+...+b_n\end{pmatrix}\right]+Bu(t)\\ y(t)=Cx(t)\end{cases} \quad (8)$$

with n *and* N represent respectively the argument number of the non-linearity and the number of linear model. For example N=1, the state representation is

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{N}\left[\begin{pmatrix}\partial_{1,1} & . & . & \partial_{1,n}\\ . & . & & \\ . & & . & \\ \partial_{1,n} & & & a_{1,i}\end{pmatrix}x(t)+\begin{pmatrix}0\\ .\\ .\\ b_i\end{pmatrix}\right]+Bu(t)\\ y(t)=Cx(t)\end{cases}$$

The state representation of the linear system (6) is:

$$\begin{cases} x(t) = A_i\, x(t) + Bu(t) + \beta_i \\ \quad y(t) = C x(t) \end{cases} \quad \forall x(t) \in X_i \tag{9}$$

With X_i is the variation domain of the state variables.

The equation (9) represent a problem at the level of the term β_i . For this reason we must eliminate this term, in order to relocate the state vector $x(t)$ by a constant x_0 .

Let's suppose now that, we have a new state variable $z(t)$ where $z(t) = x(t) - x_0$, $\dot{z}(t) = \dot{x}(t)$, the new state representation is the following:

$$\begin{cases} \dot{z}(t) = A\, z(t) + Bu(t) \\ \quad y(t) = C z(t) \end{cases} \quad \forall z(t) \in Z_i \tag{10}$$

with Z_i the variation domain of the state variables $X_i - X_{0i}$.

2.3 Constructions of uncertain multi-model

The uncertainty used in this section is a norm bounded uncertainty. This type of uncertainty is defined by an ellipse characterized by α_i *and* ρ_i :

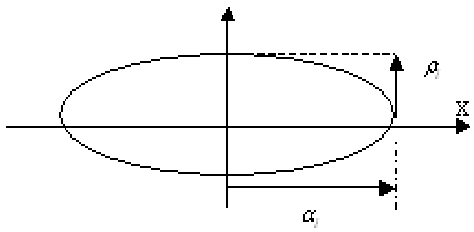

Fig2: Elliptic domain of uncertainty

The state representation of the whole linear models while considering a norm bounded uncertainty is:

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^{N} \left[\begin{pmatrix} \partial_{11} & . & . & \partial_{1n} \\ . & . & & \\ . & & . & \\ \partial_{n1} & & & a_{1,i} \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ . \\ . \\ 1 \end{pmatrix} F \begin{pmatrix} 0 & 0 & \dots & \alpha_i & \rho_i \end{pmatrix} \right] + Bu(t) \\ \qquad y(t) = Cz(t) \\ F = \left\{ F(t) \in \Re^{d.e} : F(t)^T F(t) \le I \right\} \end{cases} \tag{11}$$

3. State control for a linear systems: Multi-model approach

In this section, we present the concept of stabilisability quadratic introduced by Holot and Barmish into [2], and its application onto the stabilization of the uncertain linear systems[10]
We consider the following continue time system

$$\begin{cases} \dot{z}(t) = (A + \Delta A) z(t) + B\,u(t) \\ y = C\;z(t) \end{cases} \quad (12)$$

Where $z(t) \in \Re^{n}$ is the state, $u(t) \in \Re^{m}$ is the control input and $y(t) \in \Re^{q}$ is the measured output. Uncertainty ΔA of type norm bounded.

$$\Delta A = D\,F\,E \quad (13)$$

<u>THEOREM 1</u>: The system (12 and 13) is quadratically stabilizable by a state feedback control if and only there exist positive definite symmetric matrix $W_1 \in \Re^{n\times n}$, matrix $W_2 \in \Re^{m\times n}$ and $\varepsilon > 0$ such that :

$$\begin{bmatrix} AW_1 + W_1 A^T + BW_2 + W_2^T B_2^T + \varepsilon D_1 D_1^T & W_1 E_1^T \\ E_1 W_1 & -\varepsilon I \end{bmatrix} < 0 \quad (14)$$

the state feedback gain is $K = W_2 W_1^{-1}$

<u>Proof:</u> See[4].

Now we present the algorithm of the multi model control , we are interested more particularly in research of a séquencée order ensuring a suitable transition between an initial and a final operation point. The problem is thus to find the control laws associated for each uncertainty domain.

Algorithm

- Approximate the non-linear function by a whole of the linear functions.
- Determine the equations of each linear function $f(x)$
- Clarify the equilibrium points x_{eq}^{i} and the final point x_f .
- Specify the variation uncertain domain of each x_{eq}^{i}.
- Determine the local control law $u\;(t) = K_i\;x(t)$ by solving LMI (14)
- Repeat the step four until x_{eq}^{i} is equal x_f
- Construction the global control law by on gain scheduling form the local control law

An algorithm that joint in one hand a method to linearization of non-linear models and the other hand a method (multi-model approach) to build a feedback control law. The algorithm is shown in an example.

4. NUMERICAL EXAMPLE:

The approach is illustrated by the mean of a physical system composed of two masses (m_1 , m_2) connected by a spring. The stiffness k of this spring affects norm bounded uncertainty.

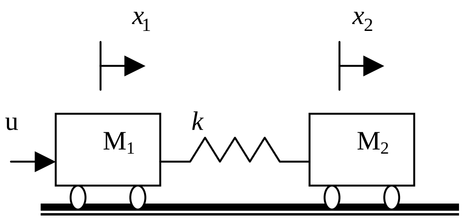

Fig3: Physical system

The state representation of this system is given by [1]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u \qquad (15)$$

$$y = x_2$$

The values of the two masses m_1 , m_2 are supposed to be known and the stiffness k is characterized by a nonlinear equation g(x)= $k(x) = e^{(x-1)}$ with $\forall x \in [0.5 \quad 4]$. The last function g(x) is mono-argument (n=1) .

The analytic expression $f(x)$ for the approximation of $g(x)$ is

$$f(x) = \sum_{k=1}^{N} [(a_{1,k} x_1 + b_1)]$$

4.1 Construction the uncertain linear model

The non-linear function $g(x) = e^{(x-1)}$ with the domain is $S = \left\{ x \in \Re^1 : 0.5 < x \leq 4 \right\}$.

The approximation HL-CPWL of a non-linear function $g(x)$ with four linear functions is represented by figure 4.

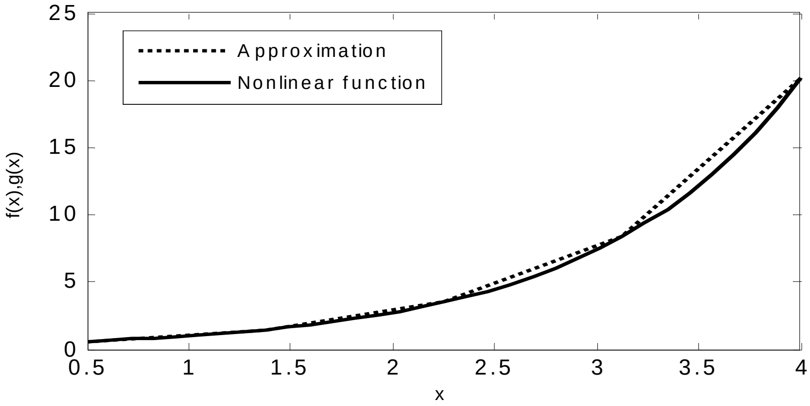

Fig4: HL-CPWL approximation of the nonlinear function with four linear functions

The norm bounded domain uncertainty of each model is represented by ellipses as we can see in the figure 5.

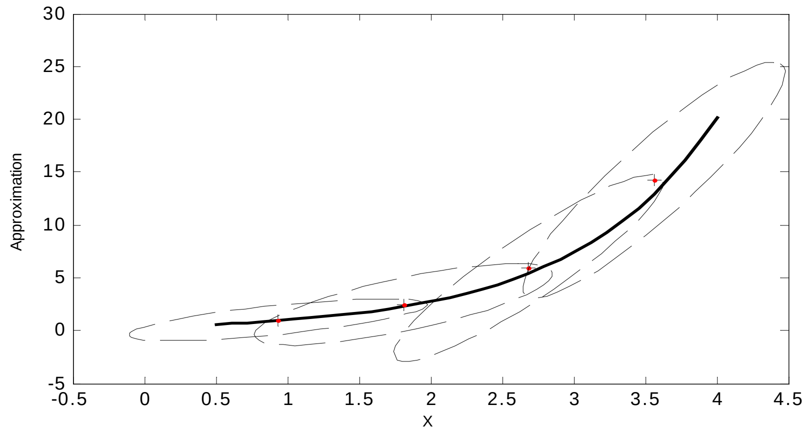

Fig5: Uncertainty elliptic domain

The new state representation of the nonlinear system is

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^{4}\left[\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_{i,1} & a_{i,1} & 0 & 0 \\ a_{i,1} & -a_{i,1} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} F(t)\begin{pmatrix} \alpha_i & \rho_i & 0 & 0 \end{pmatrix}\right) z(t)\right] + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} z(t) \end{cases}$$

	$a_{i,1}$	α_i	ρ_i
1	0.8784	0.6149	0.7
2	1.7689	1.2382	0.41
3	3.5620	2.4934	0.65
4	7.1731	5.0212	0.71

Table 1: Simulation Parameter

4.2 State control for a linear systems

The state feedback is based on the resolution of theorem1. The results of this resolution permit us to determine the feedback gain values K_i . The application of state feedback gain K_i (table 2) for each uncertain linear model makes possible to ensure local stability in its elliptic domain defined by the ellipse ξ_i characterized by α_i *et* ρ_i .

Equilibrium points:	K_i
1	[-7.0113 -13.4945 -10.8850 -5.2338]
2	[-6.5185 -23.2645 -7.4706 -6.0612]
3	[-7.0511 -32.5182 -4.9178 -7.1140]
4	[-8.4611 -37.4582 -4.4199 -5.9828]

Table2: State feedback gains

Using a gain scheduling one can now apply a law of global control. We fix , for example, like an initial condition $x\ (0) = [-1, 0, 0, 0]^T$ and a final state $x\ (\infty) = [3.1, 3.15, 4.15, 1.85]^T$. The curves (6) show the evolution of the state variables and illustrate the convergence of the trajectory which always evolves inside the ellipses of stability. It is clear that the stability is guaranteed thanks to the existence of two successive equilibrium points in the same area of stability.

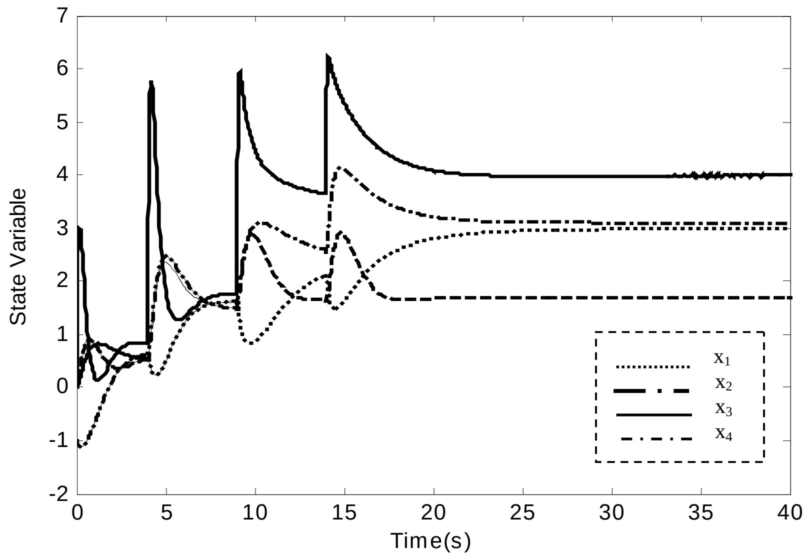

Fig6: Variation of state variables

Figure (7) illustrates the evolution of the control signal. For the latter, there are discontinuities at the instants of commutation, the amplitude of these discontinuities falls with the number of the correctors. One can think to define a continuous global law. The feedback K(t) gain is obtained by polynomial interpolation between the 4 controllers calculated previously.

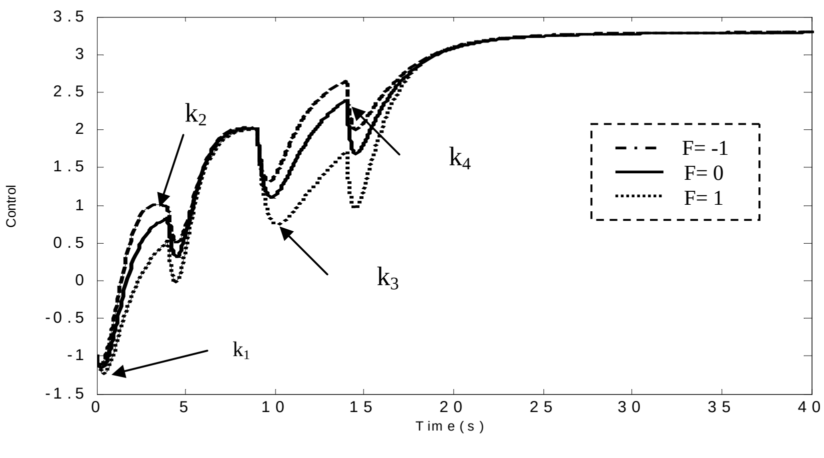

Fig7 : Gain scheduling control

5. Conclusion

The principal contribution of this article is to prove that one can use a methodology of the order multi-model to guarantee stability in a closed loop of a non-linear system. For the class of globally locally controllable non-linear systems, a numerical procedure has been presented which insure a convergent and thus stable transition from any initial stead state equilibrium to a final one. Of course, that can be achieved at the expense of high numeric computation in the case when the number of intermediate points is high which may be the case when the parameter σ has to be chosen very small. That can occur for highly non-linear systems when the local uncertainty has to be locally embed the non linearities.

However, when the system is locally controllable there always exists a non null state domain where the uncertain linear system, concerning the non-linear one, is state stablizable.

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