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A Sliding Mode Control for Multivariable Systems

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Abstract. In this paper, we propose a methodology relative to the application of a sliding mode control to linear multivariable systems. This methodology is based on the decomposition of the system in several subsystems controllable each by only one component of the input. The application of the proposed strategy in the case of a sliding mode control to the multivariable system led, in addition to the simplicity offered by the decomposition, to satisfactory simulation results in terms of the desired performances in closed loop.

Keywords: State feedback , Sliding Mode, Variable Structure, Decentralized Sliding Mode Control, SISO linear Subsystem, Multivariable.

1 Introduction

The variable structure control proved itself in regulation and tracking aspects [1, 5, 9, 12]. But this type of control was often criticized because of the chattering problem caused by the discontinuous control. Some ameliorations are proposed: the equivalent control [13], the continuous function in a band around the sliding surface [11], the generalized variable structure control [3, 9] and the higher order sliding mode control [8, 9]. These ameliorations could extend the application area to the control of hydraulic actuators [9], pneumatic actuators [9, 14],... etc. For the multivariable systems case and in the most previous work, the application of the variable structure control is carried out by supposing that the multivariable system is composed in several inter-connected subsystems and the control law is obtained by considering the interactions as external disturbances ([15] ...).

In this paper, we propose an implementation methodology of the sliding mode control on multi-input systems by assuming the state feedback control idea [4]. The contribution of the proposed methodology is highlighted by numerical simulation on an academic example.

In this paper, we initially give an overview on the sliding mode control, then we present the decomposition of a multivariable system in subsystems where each one is controllable by only one component of the input [4] and finally can lead to

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an implementation methodology of the sliding mode control on linear multivariable systems. An illustration, through an academic example, of the suggested methodology is given in section 3 of this paper.

2 Variable Structure and Sliding Mode Control

The sliding mode control constitutes the natural framework to deal with the discontinuous systems or with the systems having a variable structure.
Let's consider, first of all, the case of a continuous system defined by the differential equation: $\dot{x} = f(x,u)$ where $u \in U$ is the control, U is a convex set of R^m. We define a control law under a state feedback form and having discontinuities on surfaces defined in the state space. In the case of a monovariable system where u can take the values u^- or u^+ and in the case of a surface S (set of the state space defined by the equation $S(x) = 0$) having a codimension one, the state feedback is defined by $u = u^+$ if $S > 0$ and $u = u^-$ if $S < 0$ with $u^+ \neq u^-$.
Let's consider now a surface $S = 0$ defined in R^n. It divides the state space into two regions $S > 0$ and $S < 0$.
For a system having a behavior defined by the commutation between two vectors fields f^+ for $S > 0$ and f^- for $S < 0$, the two vectors fields f^+ and f^- point to the surface S. In this manner, any trajectory beginning its movement out of S, will join it [7, 9].
The system described above is well defined, outside of the surface S (by one of the two vectors fields f^+ or f^-); however, on S, the behavior is not defined and can be imposed.
If the system is naturally discontinuous, it is defined by a given finite set of vector fields on which the system can commutate according to the position of the state space. In the two cases of discontinuities (natural or caused by a discontinuous control), the system is described by a differential equation with discontinuous right hand. The system is well defined outside the surface, the solutions can be obtained from the classical theory of differential equations. But on the surface where the system is not defined exactly, the system behavior is to be defined according to a given direction (we note that the classical theory of the differential equations ceases to be valid, since the considered system does not verify the traditional conditions of existence of the Cauchy-Lipschitz theorem because of the discontinuity of the state vector on the commutation surface).
In the case where the surface is defined as a surface using the system output as a component of the system's state, the dynamic behavior that we seek to define on the commutation surface sliding can be considered as being the internal dynamics of controlled system corresponding to the behavior of the closed loop system.

Let x_0 a point such as $S(x_0) = 0$ where the rank of $\partial S/\partial x$ is one in x_0. It exists a neighborhood X of x_0 such as $S \in X$ which is a differential manifold (its codimension is one; S divide X in two regions characterized by the points x such as $S(x) > 0$ and $S(x) < 0$).

We suppose that $L_{f^+}S(x_0) < 0$ and $L_{f^-}S(x_0) > 0$, It exists a neighborhood of x_0 in X where the inequalities remain valid.
We suppose that the neighborhood is equal to X (for simplification).

Definition
If under the action of the vector fields, the state checks the following inequalities:

$$x \in X \Rightarrow \begin{cases} L_{f^+}S < 0 \\ L_{f^-}S > 0 \end{cases}$$

then a sliding mode exists on S (locally) [10].

Figure 1 shows the various possible configurations, and specifies the corresponding sliding mode on S [6].

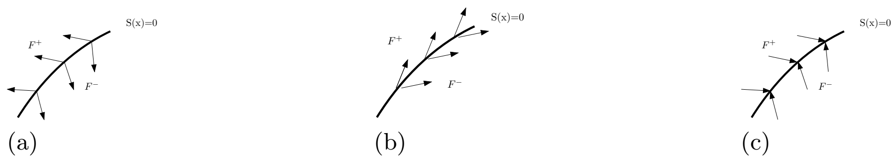


Fig. 1. Principle of the sliding mode (a - b: No Sliding Mode; c: Sliding Mode).

In literature, the previous definition is presented in several forms :

- A sliding mode exists on S if the projection of the segment generated by $f^+(x)$ and $f^-(x)$ $x \in S$ on a line crossing surface (parallel to tangent space) at the point where the sliding mode exists, contains 0 at its interior
- In the case of the continuous systems (discontinuity due to the nature of the control), a sliding mode exists on S if the system has in $x \in S \subset X$ a relative degree equal to one [10].
- A sliding mode exists on S if $S\dot{S} < 0$ under the action of the commutation law (the variety S is attractive).

The dynamics on surface S is not being determined, the problem is to define the behavior of the state once the surface is reached. The theory of the ordinary differential equations ceases to be valid because of the discontinuity of the second member (the theorem assumptions of Cauchy-Lipschitz are not filled). In literature, most researchers were interested in this problem of continuity of the differential equations solutions to deduce from it the equation which describes the dynamic behavior of the state on the discontinuous surface [2, 13]. We are interested here in the approaches of Filippov [2] and of Utkin [13].

Filippov approach [2]:
Let F a family of vector fields. The system is defined by a strategy of commutation of F elements, so that a variety S becomes attractive and such a strategy of commutation implies a sliding mode on S. According to this approach, the

sliding dynamic equation which results from the application of such a family of vector fields, is given by the vector field pertaining to the intersection of tangent space to the sliding variety with the convex envelope generated by the family $F = \{f_i(X) : i \in I\}$. The sliding dynamics in $x \in S$, is given, within the meaning of Filippov, by the intersection:

$$\overline{Conv}(F)_x \cap T_x\mathrm{S}$$

where $Conv(F)_x$ is the space generated by f^+ and f^- in x, $\overline{Conv}()$ its closing, $T_x \in S$ the tangent space to S on x. The sliding dynamic is given by [6]:

$$\begin{cases} x \in \mathrm{S} \\ f^* = \lambda\, f^+ + (1-\lambda) f^- = f^- + \lambda\,(f^+ - f^-) \end{cases}$$

and takes the following value :

$$\lambda = \frac{\langle \partial S, f^- \rangle}{\langle \partial S, (f^- - f^+) \rangle}$$

where $< .,. >$ is the scalar product.

Utkin approach [6, 13]:
In this approach, we consider systems where the discontinuity is due to the nature of the control. The considered system is :

$$\dot{x} = f(x,u) \, \text{with } u(x) \text{ discontinuous in } x$$

The sliding dynamic is the dynamic which results in replacing u by the value which makes the sliding surface invariant under the action of the field called equivalent. Let's consider the system defined by the following equations:

$$\dot{x} = f(x,u) \quad u \in U \subset \mathrm{R}$$
$$u(x) = \begin{cases} u^+(x) & \text{si } S(x) > 0 \\ u^-(x) & \text{si } S(x) < 0 \end{cases}$$

The system defined above, presents a sliding behavior. The sliding dynamic is given by :

$$f^* = f_{eq}(x) = f(x, u_{eq})$$

where u_{eq} is the equivalent control which makes the surface invariant. For $x \in S$, u_{eq} verifies the following inequality :

$$\min(u^-(x), u^+(x)) < u_{eq} < \max(u^-(x), u^+(x))$$

Remarks

- As shown in figure 2, the dynamics of Filippov and Utkin are generally different. They cannot even be "collinear". Indeed, each one corresponds to a different situation according to whether we deal with discontinuous system by nature (Filippov), or with a system which is made discontinuous by the choice of discontinuous state feedback around S (Utkin). Moreover, it is shown that they are equivalent in the case of systems linear in control.

– There are other approaches to define the dynamics on the sliding surface. One of these approaches, is the convex one which shows that sliding dynamics belongs to the intersection between tangent space and smallest closed convex space containing the vectors fields $f(x, u)$ for $u \in U$ where U is the set of admissible controls.

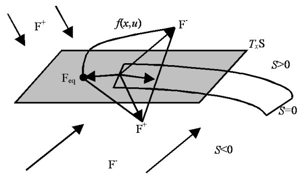


Fig. 2. Filippov's and Utkin's fields.

3 Variable structure control for the multivariable system

Let's consider a multivariable linear system described by the following equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Lx(t) \end{cases} \quad (1)$$

where :

- $x \in \mathbf{R}^n$: state vector .
- $A \in \mathbf{R}^{nxn}$: state matrix.
- $u \in \mathbf{R}^m$: input vector.
- $B \in \mathbf{R}^{nxm}$: matrix.
- $y \in \mathbf{R}^p$: output vector.
- $L \in \mathbf{R}^{pxn}$: matrix.

The sliding surface, chosen linear, is:

$$S(x) = C^T x$$

where $C \in \mathbf{R}^{mxn}$ is a parameter matrix.
Assume that the control is composed of two terms, $u = u_{eq} + \Delta u$ [9, 13]. The equivalent control u_{eq} expression is given by [9, 13]:

$$\dot{S}(x) = 0 \Rightarrow C^T \dot{x} = C^T (Ax+Bu) \Rightarrow u_{eq}(t) = -\left(C^T B\right)^{-1} C^T \left(Ax(t)\right) \quad if \left(C^T B\right)^{-1} \textit{ exists}$$

and the control is:

$$u(t) = -\left(C^T B\right)^{-1} C^T \left(A\, x(t)\right) + \Delta u \quad where \;\; \Delta u = -k\, sign\left(S(x)\right)$$

where k is a gain chosen to eliminate the perturbation effect.

So, to apply the variable structure control to a multivariable system, we must have the existence of $\left(C^T B\right)^{-1}$.

To overcome this condition, we propose in this paper a methodology based on the decomposition of the multivariable system into subsystems controllable each by one component of the input.

4 Decomposition of a multivariable system to subsystems controllable each by only one component of the input [4]

Let's consider the multivariable linear system described by equation (1).
The condition of governability of the system with regard to the inputs set is :

$$rank\left[B\ AB \cdots A^{n-1}B\right] = n \tag{2}$$

Generally, the system will be completely governable only with the action of several components of the input. We give the two possible cases.

4.1 The case where the system is completely gouvernable by one component of the input u_i:

In this case, we have :

$$rank\left[b^i\ Ab^i \cdots A^{n-1}b^i\right] = n \tag{3}$$

So, the system is completely controllable by the component u_i of the input vector u. This case is equivalent to the monovariable system case. The control is carried out on the component u_i.
The system having as input u and as output y is completely controllable by the component u_i of the input. Thus the variable structure control can be calculated as follows:
For a system of order n, the sliding surface (presumed linear according to the states of the system) is:

$$S(x) = \sum_{i=1}^{n} c_i x_i \;\; avec \; c_n = 1 \tag{4}$$

The control u must verify the sliding condition:

$$S(x)\dot{S}(x) < 0 \tag{5}$$

One of the solutions can be chosen as the basic form (component u_i)

$$u_i = -M sign(S(x)) \tag{6}$$

Then, the control vector is :

$$u = \begin{bmatrix} 0 \cdots 0 \; u_i = -M \; sign(S) \; 0 \cdots 0 \end{bmatrix}^T \tag{7}$$

Simulation example:
Let's consider the following system :

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) \end{cases}$$
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \; ; \; C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We remark here that u_1 control all modes (-1; 2; -3). So, the system is completely controllable by u_1 and it can be described, after a change of variable, by :

$$\dot{\widetilde{x}}(t) = \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\;u(t)$$
$$with \quad : \qquad \widetilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{bmatrix} \quad ; \quad \widetilde{B} = \begin{bmatrix} 0 & \mathrm{x} \\ 0 & \mathrm{x} \\ 1 & \mathrm{x} \end{bmatrix}$$

Here, we have a third order system and the sliding mode control is:

$$S = \sum_{i=1}^{3} c_i \widetilde{x}_i = c_1 \widetilde{x}_1 + c_2 \widetilde{x}_2 + \widetilde{x}_3 \quad c_1 = 10 \quad c_2 = 7$$
$$u_1 = -M\,sign(S); \quad u_2 = 0; \quad M = 20; \text{ such as } \; S\dot{S} < 0.$$
$$u = \begin{bmatrix} -M\,sign(S) \\ 0 \end{bmatrix}$$

In the error space, the surface equation becomes:

$$S = \sum_{i=1}^{3} c_i e_i = c_1 e_1 + c_2 e_2 + e_3$$

with $e_i = \widetilde{x}_i - x_{0i}$ and x_{0i} $(i = 1, 2, 3)$ the initial condition or the reference trajectory.
The simulations results for a sinusoidal reference are given in figure 3. The last one shows a relatively perfect tracking.
The evolution of the trajectory in the phase space is given by figure 4. After a certain search time, the trajectory slides until convergence towards the origin. The obtained result are completely coherent with the theoretical affirmation.

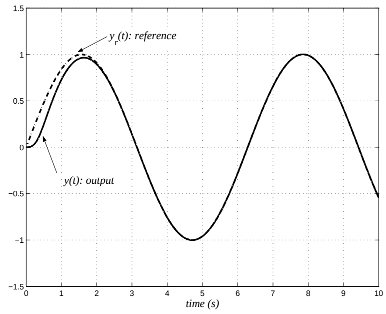


Fig. 3. Evolution of the output and the reference.

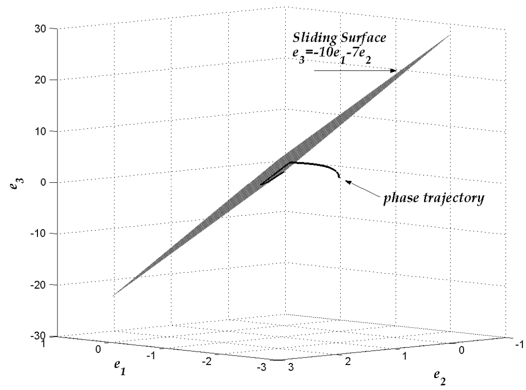


Fig. 4. The trajectory in the phase space.

4.2 The case where the system is not completely controllable by only one component of the input

Decomposition into sub-systems controllable each by one component of the input

If the system is controllable, it is always possible to decompose it into r sub-systems:

- each one is controllable by only one component of the input;
- they are treated on a hierarchical basis so that :
  - the sub-system Σ_r reacts on the sub-systems Σ_{r-1} ... Σ_2 , Σ_1.
  - the sub-system Σ_{r-1} reacts on the sub-systems Σ_{r-2} ...Σ_2 , Σ_1.
  - ...
  - the sub-system Σ_2 reacts on the sub-system Σ_1.

The diagram block of the decomposition is given by figure 5 [4].

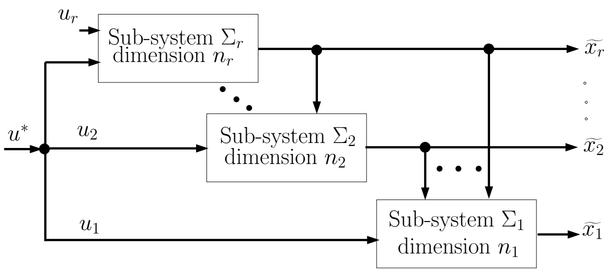


Fig. 5. System decomposition into sub-systems controllable each by one component of the input.

Refereing to the last figure, we can write :

$$u^* = \left[u_{r+1}\ u_{r+2} \cdots u_m \right]^T \qquad si \quad r < m \tag{8}$$

The state equation of the sub-system Σ_i has the following form :

$$\dot{\widetilde{x}}_i = A_{ii}\widetilde{x}_i + [A_{ii+1}\widetilde{x}_{i+1} + \cdots + A_{ir}\widetilde{x}_r] + \widetilde{b}^i u_i + \beta_i u^* \tag{9}$$

with :

$$n = \sum_{i=1}^{r} n_i$$

n and n_i represent respectively the dimension of the global system and the dimension of the sub-system Σ_i.

Variable structure control synthesis

The sub-system Σ_i is defined by the state equation (9) having the dimension n_i and the control u_i. The sliding surface, chosen linear, has the following form :

$$S_i = \sum_{i=1}^{n_i} c_i \widetilde{x}_i$$

The control can be chosen in the classical form :

$$u_i = -M_i\, sign(S_i);$$

where M_i and c_i verify the sliding condition $S_i\dot{S}_i < 0$.
Then, the global control is :

$$u = \left[u_1 \cdots u_r\ u^* \right]^T \qquad u^* \text{ is a null vector}$$

The sliding mode is carried out if and only if :

$$\forall i \in \{1, \cdots, r\} \quad S_i \dot{S}_i < 0$$

So, The sliding is carried out on the intersection of surfaces $S_i = 0$.

Simulation example
Let's consider the following system :

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) \end{cases}$$
$$A = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{bmatrix} \;;\; B = \begin{bmatrix} 0\,0 \\ 0\,1 \\ 0\,0 \\ 1\,0 \end{bmatrix} \;;\; C = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

The sub-system Σ_1 is controllable by u_1 and has a dimension equal to 2. The associated characteristic polynomial is :

$$\varphi_1(p) = p^2$$

The state equation of the sub-system Σ_1 is :

$$\dot{\widetilde{x}}_1 = A_{11}\widetilde{x}_1 + \widetilde{b}^1 u_1$$
$$A_{11} = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix} \qquad \widetilde{b}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The sub-system Σ_2 is controllable by u_2 and has a dimension equal to 2. The associated characteristic polynomial is :

$$\varphi_2(p) = p^2$$

The state equation of the sub-system Σ_2 is :

$$\dot{\widetilde{x}}_2 = A_{22}\widetilde{x}_2 + \widetilde{b}^2 u_2$$
$$A_{22} = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix} \qquad \widetilde{b}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Variable structure control synthesis:

Sub-system Σ_1 :

The sliding surface is chosen linear:

$$S_1 = \sum_{i=1}^{2} c_{1i}\widetilde{x}_{1i} = c_{11}\widetilde{x}_{11} + c_{12}\widetilde{x}_{12}$$

In the error space, the surface equation becomes:

$$S_1 = \sum_{i=1}^{2} c_{1i}e_{1i} = c_{11}e_{11} + c_{12}e_{12}$$

with $e_{1i} = \widetilde{x}_{1i} - x_{01i}$ and x_{01i} $(i = 1, 2)$ the initial condition.
The control is taken in the simplest form $u_1 = -M_1\, sign(S_1)$; M_1 and c_{1i} must verify the condition $S_1 \dot{S}_1 < 0$.
For this case, we have chosen the following values : $c_{11} = 2.5, c_{12} = 1, M_1 = 10$.

Sub-system Σ_2 :

The sliding surface is chosen linear:

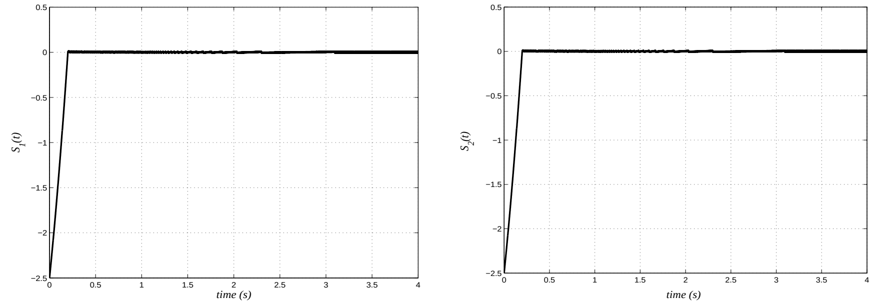
$$S_2 = \sum_{i=1}^{2} c_{2i} \widetilde{x}_{2i} = c_{21} \widetilde{x}_{21} + c_{22} \widetilde{x}_{22}$$

In the error space, the surface equation becomes:

$$S_2 = \sum_{i=1}^{2} c_{2i} e_{2i} = c_{21} e_{21} + c_{22} e_{22}$$

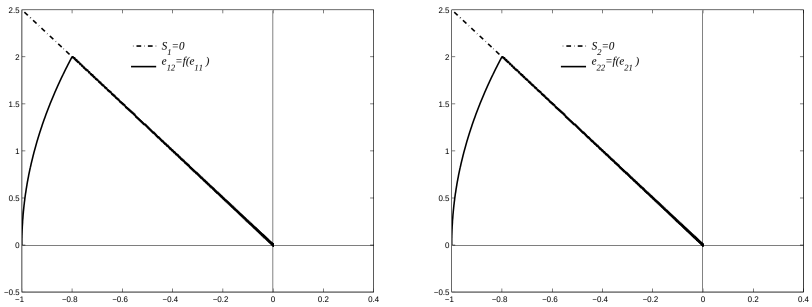
with $e_{2i} = \widetilde{x}_{2i} - x_{02i}$ and x_{02i} $(i = 1, 2)$ the initial condition.
The control is taken in the simplest form $u_2 = -M_2\, sign(S_2)$; M_2 and c_{2i} must verify the condition $S_2 \dot{S}_2 < 0$.
For the present case, we have chosen the following values: $c_{21} = 5, c_{22} = 1, M_2 = 10$.

The simulation results are given in figures 6, 7 and 8.
Figure 6 shows the evolution of the two sliding surfaces. The sliding mode is effective at the end of 0.1 s. Figure 7 represents the evolution of the phase trajectory for the two subsystems. It is noticed that the phase trajectory, once it reaches the surface, it slides on it one until reaching the origin.
Figure 8 gives an idea on the evolution of the various variables for subsystem 1 (the output, the phase plan, sliding surface and the control).
The decomposition in subsystems controllable each by only one component of the input returns the implementation of the multivariable sliding mode control which is much easier because the subsystems obtained are monovariable linear systems. The synthesis of the control is, therefore, carried out on linear monovariable systems.



(a) Evolution of $S_1(t)$ (b) Evolution of $S_2(t)$

Fig. 6. Regulation case.



(a) Sub-system Σ_1 (b) Sub-system Σ_2

Fig. 7. Phase Plane.

5 Conclusion

We have proposed a methodology of implementation of the sliding mode control on a multivariable systems by using a decomposition of the system in r subsystems controllable each by only one component of the input. Simulations presented show satisfactory results for the sliding mode control. This decomposition is valid in the case of linear systems. In the case of non linear systems, we can use the above methodology by linearizing the system around a functional point and consider the interactions as external disturbances.

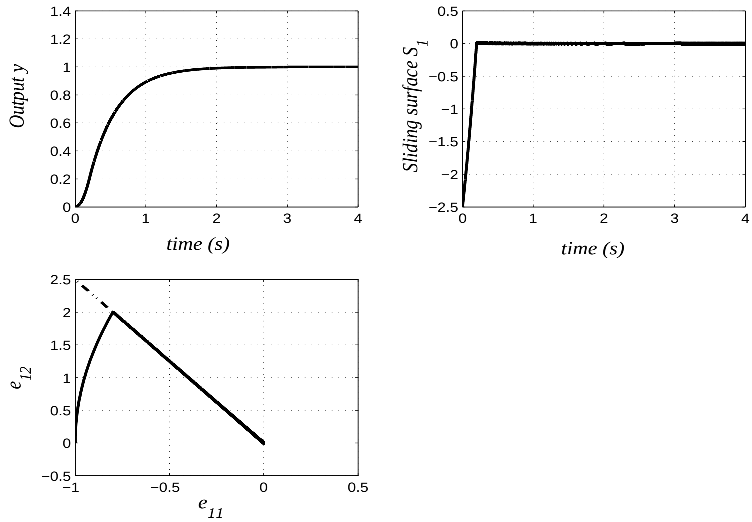


Fig. 8. Evolution of $y_1(t)$, $e_{12} = f(e_{11})$, $S_1(t)$

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