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Determination of uncertainty domain by using a subspace-based approach

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***Résumé*—In this paper we apply an identification method for a subspace system. Also we present a bounded-error approach. This approach use the iso-level curves to describe the uncertainty domain. To identify the system parameter, the least squares algorithm is used. Especially the propagator method, is introduced. To estimate the uncertainty domain, an error bounding approach is considered.**

***Index Terms*—Unertainty domain, MIMO, State-space, Propagator method, Bounded error approach.**

I. Introduction

Classically in the framework of control, the objective of estimation is to give a dynamic model of a system for control law design [1], [2]. The model is near to real system, it is necessary to set some conditions. The parameter of controller must be adjusting taking into account the model parameters uncertainty. To attend this objective, we pass through two stages. The first is to estimate a model parameters to understand the system function. The second consists in the construction of the control law. To reach this goal, we must be well-described the estimated model uncertainties.

To identify the uncertainty domain, many works are developed in this field. All this approach, have three different axes. The first works are founded on prior hypothesis about unmodeled dynamics and noise affecting the system [2]. Some approach used time-domain to represent the system parameter [3], [4] and take that the noise acting are random variables realizations [1].These development was very limit specially in case of robust control [5]. The second axe used to identify the uncertainty domain [6], [7], is based on the idea that the noise in unknown but bounded [8], [9], [10]. The problem related to this method depend how we choose the bound.

The third axe is developed to solve this kind of problem. A new approach is proposed [1], [11]. This approach is use the analysis iso-level curves to estimate the bounded-error. The Idea of this papers come from this technique. We use the state-space representations and we develop this basic idea into multi-input multi-output (MIMO). The propagator method [12], [13], will be used to identify the system parameters. This technique has some limit because it does not give access to all parameter of system but can estimate the smallest number of system parameters, for both SISO or MIMO systems.

The silhouette of this paper is presented in four sections. In Section II, we present the problem. The uncertainty domain method is determined in Section III. Then in section IV we introduce the propagator method. In Section V, we apply the new technique in numerical simulations examples. Finally, we concludes in section VI.

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II. Main problem

Consider $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ as the system matrices for coordinate state-space basis.
Defining the linear and time-invariant presentation

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \tag{1b}$$

the input $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is an ergodic, the output $\mathbf{y}(t) \in \mathbb{R}^{n_y}$, $\mathbf{v}(t) \in \mathbb{R}^{n_y}$ is the noise vectors and the state $\mathbf{x}(t) \in \mathbb{R}^{n_x}$.
In this paper, the order of the system is considered to be known *a priori* .
We make the following assumptions :

i) $\{\mathbf{v}(t)\}$ is uncorrelated with the input $\{\mathbf{u}(t)\}$,

ii) the matrix ($\mathbf{A}$ and $\mathbf{B}$) are reachable.

iii) the matrix($\mathbf{A}$ and $\mathbf{C}$) are observable.

iv) for the sake of brevity, it is assumed that $\mathbf{D} = 0$.

The model (1) is considered as output-error system [14].

Let as announce the problematic : knowing the input data $\{\mathbf{u}(t)\}_{t=1}^{N}$ and output data $\{\mathbf{y}(t)\}_{t=1}^{N}$ generated by a system (1), estimate the matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and characterize the uncertainty domain of system parameters.

III. Bounded-error method

The identification of the coefficients a_i, b_i^j and c_i^k, $i \in [0, n_x - 1]$, $j \in [1, n_u]$, $k \in [2, n_y]$ (the parameters of matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$) can be done by using the method based on propagator.
To apply this technique, the parameters is represented via linear representation. So, you will convert the state-space form into a linear regression. Then we introduce the ellipsoidal iso-level using a bounded- error method in [11]. To reach this goal, the method developed in [11] is used.

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A. Model description

Consider the state-space system (1) and we suppose

i) the model (2) order is n_x.

ii) the matrix $\mathbf{A}$ is non-derogatory.

(1) can be represented by

$$\begin{aligned}\mathbf{y}(t) = &-\left(\mathbf{a}^\top \otimes \mathbf{I}_{n_y}\right)\mathbf{y}_{n_x}(t-n_x)\\ &+\mathbf{F}\mathbf{u}_{n_x}(t-n_x)+\boldsymbol{\eta}(t) = \boldsymbol{\Phi}^\top(t)\boldsymbol{\theta}+\boldsymbol{\eta}(t) \quad (2)\end{aligned}$$

this model called an ARMAX model with

$$\boldsymbol{\Phi}^\top(t) = \left[\, -\mathbf{y}(t-n_x) \;\cdots\; -\mathbf{y}(t-1)\;\; \mathbf{u}_{n_x}(t-n_x)\otimes\mathbf{I}_{n_y} \,\right] \quad (3)$$

$$\boldsymbol{\theta} = \begin{bmatrix}\mathbf{a}\\ \mathrm{vec}(\mathbf{F})\end{bmatrix} \quad (4)$$

$$\mathbf{a}^\top = \begin{bmatrix}a_0 & a_1 & \cdots & a_{n_x-1}\end{bmatrix} \quad (5)$$

$$\mathbf{F} = \left(\mathbf{a}^\top\otimes\mathbf{I}_{n_y}\right)\mathbf{H}_{n_x}+\mathbf{C}\boldsymbol{\Delta}_{n_x}\in\mathbb{R}^{n_y\times n_x n_u} \quad (6)$$

$$\boldsymbol{\Delta}_{n_x} = \begin{bmatrix}\mathbf{A}^{n_x-1}\mathbf{B} & \cdots & \mathbf{AB} & \mathbf{B}\end{bmatrix}\in\mathbb{R}^{n_x\times n_x n_u} \quad (7)$$

and $\boldsymbol{\eta}(t)$ a noise corresponding to a filtered version of $\mathbf{v}$.

When a model is SISO, we have this special case

$$\mathbf{x}(t+1) = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}b_0^1\\ b_1^1\\ b_2^1\end{bmatrix}u(t) \quad (8a)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{x}(t) \quad (8b)$$

So, the model satisfies

$$\begin{aligned}y(t) &= -a_2 y(t-1) - a_1 y(t-2) - a_0 y(t-3) + b_0^1 u(t-1)\\ &+\left(a_2 b_0^1 + b_1^1\right)u(t-2)+\left(a_1 b_0^1 + a_2 b_1^1 + b_2^1\right)u(t-3).\end{aligned}$$

It is obvious that this model is non-linear in parameters a_i, b_i^j and c_i^k and linear in vector $\boldsymbol{\theta}$.

B. Description of uncertainty

We look for the minimum of the cost function [2, Appendix II] to estimate the parameter vector and identify the uncertainty domain of the parameters θ_i composing $\boldsymbol{\theta}$. Consider $\bar{\boldsymbol{\theta}}$ is an estimation of $\boldsymbol{\theta}$

$$J(\bar{\boldsymbol{\theta}}) = \frac{1}{2}\left(\mathbf{y}_M - \boldsymbol{\Psi}_M\bar{\boldsymbol{\theta}}\right)^\top\left(\mathbf{y}_M - \boldsymbol{\Psi}_M\bar{\boldsymbol{\theta}}\right)$$

with

$$\mathbf{y}_M = \begin{bmatrix}\mathbf{y}(1)\\ \vdots\\ \mathbf{y}(M)\end{bmatrix} \quad \boldsymbol{\Psi}_M = \begin{bmatrix}\boldsymbol{\Phi}^\top(1)\\ \vdots\\ \boldsymbol{\Phi}^\top(M)\end{bmatrix}.$$

$\mathbf{R}_{\boldsymbol{\Psi}} = \boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M$ is invertible, so that

$$\begin{aligned}J(\bar{\boldsymbol{\theta}}) &= \frac{1}{2}\mathbf{y}_M^\top\left(\mathbf{I}-\boldsymbol{\Psi}_M\left(\boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M\right)^{-1}\boldsymbol{\Psi}_M^\top\right)\mathbf{y}_M\\ &+\frac{1}{2}\left(\bar{\boldsymbol{\theta}}-\boldsymbol{\theta}_{ls}\right)^\top\boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M\left(\bar{\boldsymbol{\theta}}-\boldsymbol{\theta}_{ls}\right)\\ &= J_{min}+\frac{1}{2}d\bar{\boldsymbol{\theta}}^\top\mathbf{R}_{\boldsymbol{\Psi}}d\bar{\boldsymbol{\theta}} \quad (9)\end{aligned}$$

with $\boldsymbol{\theta}_{ls} = \left(\boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M\right)^{-1}\boldsymbol{\Psi}_M^\top\mathbf{y}_M$ and $d\bar{\boldsymbol{\theta}}$ the variation of the estimate $\bar{\boldsymbol{\theta}}$ around $\boldsymbol{\theta}_{ls}$. The unique minimum of the function $J(\bar{\boldsymbol{\theta}})$ is $\boldsymbol{\theta}_{ls}$.

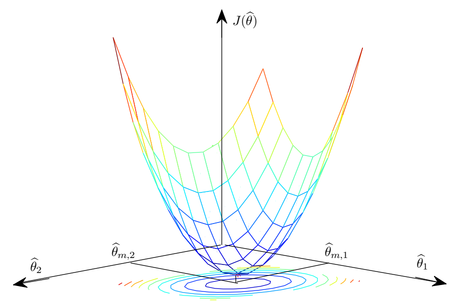

FIGURE 1. Iso-criterion curve.

Fig. 1 is given in two dimensions by consider $\widehat{\boldsymbol{\theta}}^T = \begin{bmatrix}\widehat{\boldsymbol{\theta}}_1 & \widehat{\boldsymbol{\theta}}_2\end{bmatrix}$. This figure show that the only minimum of the criteria $J(\bar{\boldsymbol{\theta}})$ is $\boldsymbol{\theta}$ the vector of estimated parameters. If we consider $V(\bar{\boldsymbol{\theta}}) = J(\bar{\boldsymbol{\theta}}) - J_{min}$, it is clear that

$$V(\bar{\boldsymbol{\theta}}) = \frac{1}{2}d\bar{\boldsymbol{\theta}}^\top\mathbf{R}_{\boldsymbol{\Psi}}d\bar{\boldsymbol{\theta}} \quad (10)$$

is a form of ellipsoid shape centered in $\boldsymbol{\theta}_{ls}$ whose directions are specified by $\mathbf{R}_{\boldsymbol{\Psi}}$.

As we say before the goal of this paper is to calculate the uncertainty domain such that the parameter$\boldsymbol{\theta}$ is in $\mathcal{D}$. Here, $\mathcal{D}$ will have ellipsoidal shape which belong on a level $J_{\mathcal{D}}$ of user. Under the Gaussian case assumptions, it is clear that [2]

$$\frac{1}{2}\left(\boldsymbol{\theta}_{ls}-\boldsymbol{\theta}\right)^\top\boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M\left(\boldsymbol{\theta}_{ls}-\boldsymbol{\theta}\right) = n^2\sigma^2$$

where σ^2 is the variance of noise and $n \in \mathbb{R}^+$. This resemblance give rise to choose the level $J_{\mathcal{D}}$ as follows

$$J_{\mathcal{D}} - J_{min} = n^2\sigma^2.$$

So, if we know σ^2 the problem is solved and we can easily get the uncertainty domain.

$$J_{min} = \frac{1}{2}\mathbf{v}^\top\left(\mathbf{I}-\boldsymbol{\Psi}_M\left(\boldsymbol{\Psi}_M^\top\boldsymbol{\Psi}_M\right)^{-1}\boldsymbol{\Psi}_M^\top\right)\mathbf{v} = \frac{1}{2}\boldsymbol{\varepsilon}^\top\boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon} = \mathbf{y} - \hat{\mathbf{y}}(\bar{\boldsymbol{\theta}})$ are the residuals. We consider ℓ as

$$\ell = \max\left\{|\boldsymbol{\varepsilon}(t)|\right\}.$$

Let us introduce $\boldsymbol{\theta}_{ss}$ as the vector of parameters, *i.e.*

$$\begin{aligned}\boldsymbol{\theta}_{ss} = \big[-a_0 \;\; \cdots \;\; -a_{n_x-1} \;\; b_0^1 \;\; \cdots \;\; b_0^{n_u} \;\; b_1^1 \;\; \cdots\\ b_{n_x-1}^{n_u} \;\; c_0^2 \;\; \cdots \;\; c_0^{n_y} \;\; c_1^2 \;\; \cdots \;\; c_{n_x-1}^{n_y}\big].\end{aligned}$$

We know that $\boldsymbol{\theta}_{ss}$ and $\boldsymbol{\theta}$ can be related via a mapping f known *a priori* (see Eq. (4)), *i.e.* $\boldsymbol{\theta} = f(\boldsymbol{\theta}_{ss})$. Employing the expansion of Taylor series, we can write

$$d\boldsymbol{\theta} \approx \left(\frac{\partial f}{\partial\boldsymbol{\theta}_{ss}}\right)_{\boldsymbol{\theta}_{ss}=\hat{\boldsymbol{\theta}}_{ss}} d\boldsymbol{\theta}_{ss}$$

where $\hat{\boldsymbol{\theta}}_{ss}$ is the estimated parameters vector. Mixing this equation with Eq. (9)

$$J \approx J_{min} + d\boldsymbol{\theta}_{ss}^{\top}\mathbf{R}d\boldsymbol{\theta}_{ss}$$

with

$$\mathbf{R} = \left(\frac{\partial f}{\partial \boldsymbol{\theta}_{ss}}\right)^{\top}_{\boldsymbol{\theta}_{ss}=\hat{\boldsymbol{\theta}}_{ss}} \mathbf{R}_{\boldsymbol{\Psi}} \left(\frac{\partial f}{\partial \boldsymbol{\theta}_{ss}}\right)_{\boldsymbol{\theta}_{ss}=\hat{\boldsymbol{\theta}}_{ss}}.$$

C. Case study

The state-space form (8) is used to estimate the model. Then,

$$\begin{aligned} \boldsymbol{\theta}_{ss} &= \begin{bmatrix} -a_0 & -a_1 & -a_2 & b_0^1 & b_1^1 & b_2^1 \end{bmatrix} \\ \boldsymbol{\theta} &= \begin{bmatrix} a_0 & a_1 & a_2 & a_1 b_0^1 + a_2 b_1^1 + b_2^1 & a_2 b_0^1 + b_1^1 & b_0^1 \end{bmatrix}. \end{aligned}$$

Furthermore,

$$\left(\frac{\partial f}{\partial \boldsymbol{\theta}_{ss}}\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -b_0^1 & -b_1^1 & a_1 & a_2 & 1 \\ 0 & 0 & -b_0^1 & a_2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

IV. IDENTIFICATION METHOD : THE PROPAGATOR

A. Main idea

The most interesting point of subspace-based identification methods is the estimation of parameters can be done from I/O data [15], [16], [17], [18], [19].
Most of the SMI algorithms are based on a three-step procedure [14] :
i) concatenation of the I/O data into some over-parameterized block data matrices according to some past and future horizons,
ii) "compression" of the I/O information using orthogonal or oblique projections (via RQ factorization) and selection of particular subspaces,
iii) model reduction and order estimation using singular value decomposition (SVD).
This identification procedure leads to minimal fully-parametrized state-space realizations. In fact, the method explained hereafter avoids the use of the SVD and fixes the state-space model basis during the observability subspace estimation.

In this method it is important to estimation of the extended observability matrix column space

$$\boldsymbol{\Gamma}_f(\mathbf{A},\mathbf{C}) = \boldsymbol{\Gamma}_f = \begin{bmatrix} \mathbf{C}^{\top} & \cdots & (\mathbf{C}\mathbf{A}^{f-1})^{\top} \end{bmatrix}^{\top}$$

f is an integer ($f \geq n_x$, user-defined).
It is straightforward to take out the state-space matrices from $\boldsymbol{\Gamma}_f$. Let start by the following relation [14]

$$\mathbf{Y}_f(t) = \boldsymbol{\Gamma}_f\mathbf{X}(t) + \mathbf{H}_f\mathbf{U}_f(t) + \mathbf{V}_f(t) \tag{11}$$

where $\mathbf{Y}_f(t)$ and $\mathbf{V}_f(t)$ have the same structure as $\mathbf{U}_f(t)$.

$$\begin{aligned} \mathbf{X}(t) &= \begin{bmatrix} \mathbf{x}(t) & \cdots & \mathbf{x}(t+M-1) \end{bmatrix} \\ \mathbf{u}_f(t) &= \begin{bmatrix} \mathbf{u}^{\top}(t) & \cdots & \mathbf{u}^{\top}(t+f-1) \end{bmatrix}^{\top} \\ \mathbf{U}_f(t) &= \begin{bmatrix} \mathbf{u}_f(t) & \cdots & \mathbf{u}_f(t+M-1) \end{bmatrix} \end{aligned}$$

M is defined in a way compatible with the full number of I/O measurements N and $\mathbf{H}_f$ is a Toeplitz matrix.

B. Procedure of identification

1) MISO system : Considering a system with one output and multi input. In this part we assume that $\mathbf{V}_f(t) = 0$. Applying an orthogonal projection to the equation (11), we find

$$\mathbf{Y}_f\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp} = \boldsymbol{\Gamma}_f\mathbf{X}\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp} \tag{12}$$

The system is observable and $\boldsymbol{\Gamma}_f$ can be written as

$$\boldsymbol{\Gamma}_f = \begin{bmatrix} \boldsymbol{\Gamma}_{n_x} \\ \boldsymbol{\Gamma}_{n_x}^c \end{bmatrix} \begin{matrix} \}\mathbb{R}^{n_x \times n_x} \\ \}\mathbb{R}^{(f-n_x)\times n_x} \end{matrix} \tag{13}$$

From (13), it is clear that $\boldsymbol{\Gamma}_f$ is devided into two part : $\boldsymbol{\Gamma}_{n_x}$ which represent the first n_x row of $\boldsymbol{\Gamma}_f$ and $\boldsymbol{\Gamma}_{n_x}^c$ which is the rest of the matrix $\boldsymbol{\Gamma}_f$.
$\boldsymbol{\Gamma}_{n_x}$ is square and non singular. We define the propagator $\mathbf{P} \in \mathbb{R}^{n_x \times n_y - n_x}$ as

$$\boldsymbol{\Gamma}_{n_x}^c = \mathbf{P}\boldsymbol{\Gamma}_{n_x}. \tag{14}$$

Using (12) it is clear that

$$\mathbf{Y}_f\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} \boldsymbol{\Gamma}_{n_x}\mathbf{X}\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp}.$$

Now, introducing $\tilde{\mathbf{X}} = \mathbf{T}\mathbf{X}$ with $\tilde{\mathbf{X}} = \mathbf{T}\mathbf{X}$ is transformation matrix

$$\mathbf{Y}_f\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{P} \end{bmatrix} \tilde{\mathbf{X}}\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp}. \tag{15}$$

To estimate the propagator $\mathbf{P}$, we introduce the matrix $\mathbf{E}$

$$\mathbf{E} = \mathbf{Y}_f\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} \mathbf{T}\mathbf{X}\boldsymbol{\Pi}_{\mathbf{U}_f}^{\perp}. \tag{16}$$

So P can be estimated by minimizing the criteria

$$\|\mathbf{E}_2 - \mathbf{P}\mathbf{E}_1\|_F^2. \tag{17}$$

The matrix $\boldsymbol{\Gamma}_f$ is written

$$\boldsymbol{\Upsilon}_f(2:n_x+1,1) = \begin{bmatrix} c\boldsymbol{\mathcal{A}} \\ c\boldsymbol{\mathcal{A}}^2 \\ \vdots \\ c\boldsymbol{\mathcal{A}}^{n_x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{A}}(1,:) \\ \boldsymbol{\mathcal{A}}^2(1,:) \\ \vdots \\ \boldsymbol{\mathcal{A}}^{n_x}(1,:) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \mathbf{P}(1,1) & \mathbf{P}(1,2) & \mathbf{P}(1,3) & \cdots & \mathbf{P}(1,n_x) \end{bmatrix}. \tag{18}$$

we define

$$\boldsymbol{\mathcal{A}}^{n_x} = -a_{n_x-1}\boldsymbol{\mathcal{A}}^{n_x-1} - a_{n_x-2}\boldsymbol{\mathcal{A}}^{n_x-2} - \cdots - a_0\mathbf{I}_{n_x}$$

Applying the Caley Hamilton formula

$$\begin{aligned} \boldsymbol{c}\boldsymbol{\mathcal{A}}^{n_x} &= \begin{bmatrix}\mathbf{P}(1,1) & \mathbf{P}(1,2) & \cdots & \mathbf{P}(1,n)\end{bmatrix} \\ &= -a_{n_x-1}\boldsymbol{c}\boldsymbol{\mathcal{A}}^{n_x-1} - a_{n_x-2}\boldsymbol{c}\boldsymbol{\mathcal{A}}^{n_x-2} - \cdots - a_0\boldsymbol{c} \\ &= -a_{n_x-1}\boldsymbol{\mathcal{A}}^{n_x-1}(1,:) - a_{n_x-2}\boldsymbol{\mathcal{A}}^{n_x-2}(1,:) - \cdots - a_0\boldsymbol{c} \\ &= \begin{bmatrix}-a_0 & -a_1 & \cdots & -a_{n_x-1}\end{bmatrix}. \end{aligned}$$

with

$$\boldsymbol{c} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}. \tag{19}$$

In similar way you can write

$$\begin{aligned} \boldsymbol{\mathcal{A}}^2(1,:) &= \boldsymbol{\mathcal{A}}(1,:)\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(2,:) \\ \boldsymbol{\mathcal{A}}^3(1,:) &= \boldsymbol{\mathcal{A}}(1,:)\boldsymbol{\mathcal{A}}^2 = \boldsymbol{\mathcal{A}}^2(2,:) = \boldsymbol{\mathcal{A}}(2,:)\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(3,:) \\ &\vdots \\ \boldsymbol{\mathcal{A}}^{n_x-1}(1,:) &= \cdots = \boldsymbol{\mathcal{A}}(n_x-1,:), \end{aligned}$$

$\mathbf{A}$ is written

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n_x-1} \end{bmatrix}.$$

We know $\mathbf{A}$ and c, so the estimation of the matrix $\mathbf{B}$ is founded by

$$\widehat{\boldsymbol{\mathcal{B}}} = \arg\min_{\mathcal{B}} \sum_t \left[\mathbf{y}(t) - (\sum_{k=0}^{t-1} \mathbf{u}(k) \otimes \widehat{\boldsymbol{c}}\widehat{\boldsymbol{\mathcal{A}}}^{t-k-1})\text{vec}(\boldsymbol{\mathcal{B}})\right]$$

2) MIMO system : The first step of the propagator method is similar to the one applied in classic subspace-based techniques, *i.e.* the compression of the I/O information. More precisely, using an orthogonal projection of the output $\mathbf{Y}_f$ into the complement of input $\mathbf{U}_f$, the forced response is removed and the problem of the unknown matrix $\mathbf{H}_f$ is solved

$$\mathbf{Y}_f\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} = \mathbf{\Gamma}_f\mathbf{X}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} \tag{20}$$

with $\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} = \mathbf{I}_{n_u} - \mathbf{U}_f^{\top}(\mathbf{U}_f\mathbf{U}_f^{\top})^{-1}\mathbf{U}_f$ in order to find the matrix $\mathbf{H}_f$ problem.

The next step is how to estimate the parameters of model. The difficulty is the selection of these n_x rows. More particularly, it is based on the observation (see [20, Lemma 1] for a proof).

The dynamics of system are consider from an auxiliary output $\mathbf{y}_a(t) = \sum_{i=1}^{n_y} \kappa_i y_i(t)$ where y_i is the system output. When κ_j, $j \in [1, n_y]$, are randomly choosed, we define a special matrix $\mathbf{K}$

$$\mathbf{K} = \begin{bmatrix} \kappa_1 & \kappa_2 & \cdots & \kappa_{n_y} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n_y \times n_y}$$

can be introduced to substitute. Then, using this transformation, Eq. (20) becomes

$$\bar{\mathbf{Y}}_f\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} = \bar{\mathbf{\Gamma}}_f\mathbf{X}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} \tag{21}$$

with

$$\begin{aligned} \bar{\mathbf{\Gamma}}_f &= \mathbf{\Gamma}_f\left(\mathbf{A}, \bar{\mathbf{C}}\right) \\ \bar{\mathbf{C}} &= \begin{bmatrix} \bar{\mathbf{c}}_1 \\ \bar{\mathbf{c}}_2 \\ \vdots \\ \bar{\mathbf{c}}_{n_y} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n_y} \kappa_j \mathbf{c}_j \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_{n_y} \end{bmatrix} \end{aligned}$$

where the rows of $\mathbf{C}$ are represented by $\mathbf{c}_j$, $j \in [1, n_y]$. $\bar{\mathbf{\Gamma}}_f$ has all column rank, a permutation matrix $\mathbf{S}$ (see [12] for its construction) can be used to reorganize the rows of $\bar{\mathbf{\Gamma}}_f$

$$\mathbf{S}\bar{\mathbf{\Gamma}}_f = \begin{bmatrix} \mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right) \\ \mathbf{\Gamma}_{f-n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right)\mathbf{A}^{n_x+1} \\ \mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \mathbf{c}_2\right) \\ \vdots \\ \mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \mathbf{c}_{n_y}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right) \\ \mathbf{\Gamma}^c_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right) \end{bmatrix}.$$

By construction, $\text{rank}\{\mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right)\} = n_x$. Hence, $\mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right)$ can be used as a similarity transformation $\mathbf{T}$. Thus a special matrix $\mathbf{P} \in \mathbb{R}^{n_y f - n_x \times n_x}$ exists and called the propagator [21] :

$$\mathbf{\Gamma}^c_{n_x} = \mathbf{P}\mathbf{\Gamma}_{n_x}.$$

Eq. (21) becomes

$$\begin{aligned} \mathbf{S}\bar{\mathbf{Y}}_f\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} &= \mathbf{S}\bar{\mathbf{\Gamma}}_f\mathbf{X}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} \\ &= \begin{bmatrix}\mathbf{I}_{n_x} \\ \mathbf{P}\end{bmatrix}\mathbf{\Gamma}_{n_x}\left(\mathbf{A}, \bar{\mathbf{c}}_1\right)\mathbf{X}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} \\ &= \begin{bmatrix}\mathbf{I}_{n_x} \\ \mathbf{P}\end{bmatrix}\tilde{\mathbf{X}}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} \end{aligned}$$

with $\tilde{\mathbf{X}} = \mathbf{T}\mathbf{X}$. In case that the propagator can be estimated from input and output signals, this relation proof that the observability of the system subspace is presented in a special basis . If the propagator $\mathbf{P}$ is known, we can write

$$\mathbf{\Upsilon}_f = \mathbf{S}^{\top}\begin{bmatrix}\mathbf{I}_{n_x} \\ \mathbf{P}\end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{C}} \\ \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}} \\ \vdots \\ \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}}^{f-1} \end{bmatrix}$$

with $\boldsymbol{\mathcal{C}} = \mathbf{C}\mathbf{T}^{-1}$ and $\boldsymbol{\mathcal{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$. Now for the estimation of $\mathbf{P}$, we introduce the matrix $\mathbf{Z}$ as follows

$$\mathbf{Z} = \mathbf{S}\bar{\mathbf{Y}}_f\mathbf{\Pi}^{\perp}_{\mathbf{U}_f} = \begin{bmatrix}\mathbf{Z}_1 \\ \mathbf{Z}_2\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{n_x} \\ \mathbf{P}\end{bmatrix}\tilde{\mathbf{X}}\mathbf{\Pi}^{\perp}_{\mathbf{U}_f}.$$

Using all this assumptions, it is clear that a good estimate of $\mathbf{P}$ can be got by calculate the minimum $\|\mathbf{Z}_2 - \mathbf{P}\mathbf{Z}_1\|_F^2$. Then, considering the Cayley Hamilton formula [15], we find that

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} (2:n_x+1,:) \tag{22}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n_x-1} \end{bmatrix} \tag{23}$$

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

$$\boldsymbol{c}_j = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} ((j-1)f+1,:) \text{ for } j \in [2, n_y]\,. \tag{25}$$

where a_j, $j \in [0, n_x - 1]$, are the coefficients of the characteristic polynomial of $\mathbf{A}$. The matrix $\boldsymbol{\mathcal{B}}$ can be calculated by linear regression [14], if $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$ are known. Thanks to this method, the number of parameters is $n_x(n_u + n_y)$ [22]. This is will make the estimation of uncertainty areas more easier.

V. Simulation example

The following state-space matrices are used

$$\begin{bmatrix} x^1(t+1) \\ x^2(t+1) \\ x^3(t+1) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \end{bmatrix} + \mathbf{B} \begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix}$$

$$\begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} = \mathbf{C} \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \end{bmatrix} + \mathbf{v}(t)$$

with

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 & -1 \\ 1 & 0.3 & 5 \\ -2 & -0.4 & -0.6 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{C} = \begin{bmatrix} 5 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}.$$

These matrices can be rewritten as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1 & 0.2 & -0.1 \end{bmatrix}$$

$$\text{B} = \begin{bmatrix} 7.3 & 0.2 \\ -0.3 & -6.9 \\ -14.6 & -2.0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1.0 & 0 & 0 \\ 1.0 & -0.3 & 0.6 \end{bmatrix}.$$

A 1000 Monte Carlo realization is used.

The noise ratio equals $20dB$. We choose PRBS=1000 for the input signal.
For the identification of vector parameters we use the propagator technique.

In first step you will plot the realistic uncertainty domains. This domains are plotted for a number of points equal to 2×30.

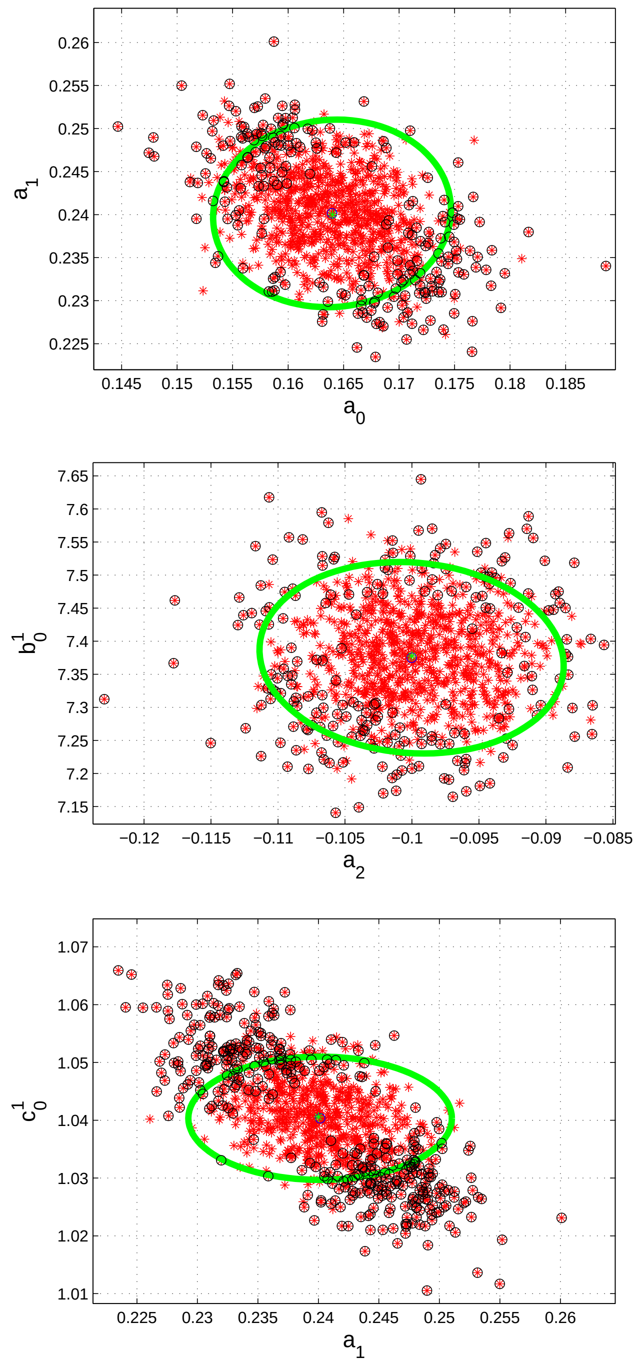

FIGURE 2. Uncertainty domain for 2×30 points

Figure 3 present the parameter of system in red color ($\times$), the mean value in black color ($+$) and the estimated parameters in blue color ($*$) calculated in 1000 realizations of the Monte Carlo.
to evaluate this method, we measure the failure rate. This rate is in percentage which present the percentage of system parameters outside of the ellipsoid domain.
The failure ratio equals 1.7 % for $(-a_0, -a_1)$ and 2.4 % for $(-a_2, b_0^1)$.
Failure ratio measures.

Parameters	$(-a_0, -a_1)$	$(-a_2, b_0^1)$	$(-a_1, c_0^1)$
Failure ratio	1.8 %	2.4 %	8.4 %

Table V introduce the failure rates for each case. this mesure show how this method is reliable

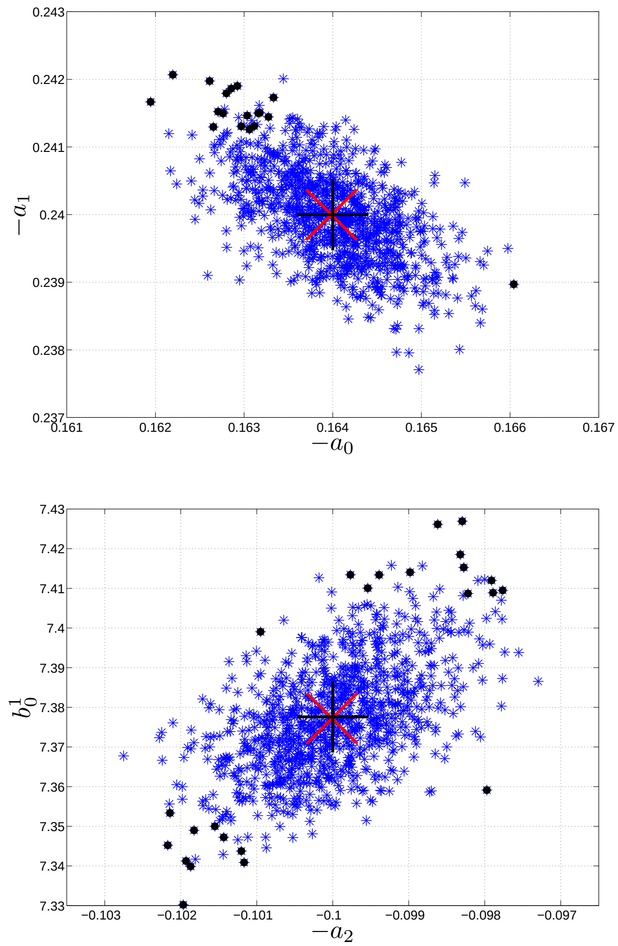

FIGURE 3. The cost function $J(\bar{\boldsymbol{\theta}})$ presentation.

VI. CLOSURE

We represent a method to identify the uncertainty domain by using the bounded-error approch. Two steps are used. the first one is to identify the model parameter by using the propagator method. This method is adequate for multi input and multi output system which is represente in state-space model. The second step is to quantify the uncertainty domain. This domains are obtained by minimize an iso-criteria to obtain an ellipsoid shape centred in estimated parameter. The efficiency of the proposed method is show through an experimental simulation. A failure rate is calculate to show the performance of results.

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