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Robust Reduced Order $\mathcal{H}_\infty$ Control via an Unbiased Observer

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Résumé. *This paper investigates the robust $\mathcal{H}_\infty$ reduced order observer-based control. The control gain is designed using standard $\mathcal{H}_\infty$ techniques. This gain is then used to resolve unbiasedness condition on the estimation error. Finally, the observer matrices are derived through the computation of a unique gain by means of Linear Matrix Inequalities.*

Keywords. *Observer-based control, Reduced-order controller, $\mathcal{H}_\infty$ optimization, LMI.*

1 Introduction

The observer-based control is usually applied (see [1–8]) when we do not have access to all the states of a system. In the case of linear systems without uncertainty, it is well established that the observer-based control problem can be resolved into two separate problems: the observer synthesis and the control law design.

Thus, this separation principle permits to decrease the computation time and to reduce the complexity of the synthesis problem because this one is divided into two separate subproblems.

But in many practical situations, there are uncertainties which affect the nominal system. Here, we will consider the observer-based control problem for a system subjected to norm-bounded uncertainties as in [9–14].

In this paper, a method is proposed to design a robust reduced order control law for uncertain systems into two steps. First, we search for a linear control law which ensures an $\mathcal{H}_\infty$ performance. Second a functional filtering techniques developed in [15, 16] is used to determine the observer-controller matrices. We prove that the obtained observer-based controller guarantees the quadratic stability and an $\mathcal{H}_\infty$ performance. We use a functional filter which estimates the control law without estimating the all state of the system contrary to "standard" observer-based control approach (see [1]). Our approach allows to reduce the order of the controller since the designed functional filter is of the same order than the functional to be estimated, *i.e.* the dimension of the control input $u(t)$.

In the sequel, $A^\dagger$ is a generalized inverse of matrix A satisfying $A = AA^\dagger A$ [17].

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2 Preliminary results

Let us consider the following uncertain system

$$\dot{x} = (A + \Delta_A(t))\, x + (B + \Delta_B(t))\, w \tag{1a}$$
$$z = Cx + Dw. \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^q$ is the output, and $w(t) \in \mathbb{R}^r$ is the disturbance vector.

The matrices $\Delta_A(t)$ and $\Delta_B(t)$ represent the parametric uncertainties and satisfy the following relation

$$\begin{bmatrix}\Delta_A(t) \ \Delta_B(t)\end{bmatrix} = M\Delta(t) \begin{bmatrix}N_x \ N_w\end{bmatrix} \tag{2}$$

with $\Delta^T(t)\Delta(t) \leqslant I_k$ and $\Delta(t) \in \mathbb{R}^{\ell \times k}$.

In this paper the usual $\mathcal{H}_\infty$ performance index J_{zw} will be considered i.e.

$$J_{zw} = \int_0^\infty \left(z^T z - \gamma^2 w^T w\right) \mathrm{d}\, t \tag{3}$$

for a given $\gamma > 0$.

2.1 First formulation of the preliminary result

The following lemma is given to ensure the stability and the $\mathcal{H}_\infty$ perfomance of system (1) and is used to prove theorem 2 in the sequel.

Lemma 1. *The uncertain system* (1) *is quadratically stable and satisfies the $\mathcal{H}_\infty$ performance $J_{zw} < 0$ for a given $\gamma > 0$ if there exist matrices $P = P^T > 0$, G and a real $\mu > 0$ such that*

$$\begin{bmatrix} GA + A^TG^T + \mu N_x^T N_x & P - G + A^T G^T & GB + \mu N_x^T N_w & C^T & GM \\ P - G^T + GA & -G - G^T & GB & 0 & GM \\ B^T G^T + \mu N_w^T N_x & B^T G^T & -\gamma I + \mu N_w^T N_w & D^T & 0 \\ C & 0 & D & -\gamma I & 0 \\ M^T G^T & M^T G^T & 0 & 0 & -\mu I \end{bmatrix} < 0. \tag{4}$$

■

Note that if LMI (4) is feasible, then $G + G^T > 0$, so G is nonsingular.

Proof. As in [18–20], let us consider the following descriptor system (using the relation $\phi = \dot{x}$)

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ A_\Delta & -I_n \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ B_\Delta \end{bmatrix} w \tag{5a}$$
$$z = \begin{bmatrix} C \ 0 \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} + Dw \tag{5b}$$

where

$$A_\Delta = (A + \Delta_A(t)) \quad \text{and} \quad B_\Delta = (B + \Delta_B(t))\,. \tag{6}$$

Consider the Lyapunov function candidate given by ($P = P^T > 0$))

$$V(x) = x^T P x = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ G^T & G^T \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{7}$$

where G is a given matrix with appropriate dimensions. Then

$$\dot{V}(x) = 2 \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} P & G \\ 0 & G \end{bmatrix} \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} P & G \\ 0 & G \end{bmatrix} \begin{bmatrix} \dot{x} \\ -\dot{x} + A_\Delta x + B_\Delta w \end{bmatrix}. \tag{8}$$

Using (5) into (8) and from the fact that $V(0) = 0$ and $V(\infty) \geqslant 0$, we have the following inequality

$$J_{zw} \leqslant \int_0^\infty \left(z^T z - \gamma^2 w^T w + \dot{V}(x) \right) \mathrm{d}\,t \tag{9}$$

where J_{zw} is the usual $\mathcal{H}_\infty$ performance index.

Inequality (9) is equivalent to (see (8))

$$J_{zw} \leqslant \int_0^\infty \left(\begin{bmatrix} x^T & \dot{x}^T & w^T \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & -\gamma^2 I + D^T D \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ w \end{bmatrix} \right) \mathrm{d}\,t \tag{10}$$

where

$$\Theta_{11} = \begin{bmatrix} 0 & I \\ A_\Delta & -I \end{bmatrix}^T \begin{bmatrix} P & 0 \\ G^T & G^T \end{bmatrix} + \begin{bmatrix} P & G \\ 0 & G \end{bmatrix} \begin{bmatrix} 0 & I \\ A_\Delta & -I \end{bmatrix}$$

$$\Theta_{12} = \begin{bmatrix} P & G \\ 0 & G \end{bmatrix} \begin{bmatrix} 0 \\ B_\Delta \end{bmatrix} + \begin{bmatrix} C^T D \\ 0 \end{bmatrix}$$

So, if the relation

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 \\ \Theta_{12}^T & -\gamma^2 I & D^T \\ 0 & D & -I_q \end{bmatrix} < 0 \tag{11}$$

holds, then $J_{zw} < 0$ ($\dot{V}(x) < 0$ also).

Moreover using the Schur lemma, inequality (11) is equivalent to

$$\begin{bmatrix} GA_\Delta + A_\Delta^T G^T & P - G + A_\Delta^T G^T & GB_\Delta & C^T \\ P - G^T + GA_\Delta & -G - G^T & GB_\Delta & 0 \\ B_\Delta^T G^T & B_\Delta^T G^T & -\gamma I & D^T \\ C & 0 & D & -\gamma I \end{bmatrix} < 0$$

which can be written as

$$\begin{bmatrix} GA + A^T G^T & P - G + A^T G^T & GB & C^T \\ P - G^T + GA & -G - G^T & GB & 0 \\ B^T G^T & B^T G^T & -\gamma I & D^T \\ C & 0 & D & -\gamma I \end{bmatrix} + H_1 \Delta(t) H_2 + H_2^T \Delta^T(t) H_1^T < 0 \tag{12}$$

with $H_1 = \left[\, (GM)^T \;\; (GM)^T \;\; 0 \;\; 0 \,\right]^T$, $H_2 = [\, N_x \;\; 0 \;\; N_w \;\; 0 \,]$.

Finally it is well-known that if there exists $\mu > 0$ such that

$$\begin{bmatrix} GA + A^T G^T & P - G + A^T G^T & GB & C^T \\ P - G^T + GA & -G - G^T & GB & 0 \\ B^T G^T & B^T G^T & -\gamma I & D^T \\ C & 0 & D & -\gamma I \end{bmatrix} + \mu^{-1} H_1 H_1^T + \mu H_2^T H_2 < 0 \quad (13)$$

then inequality (12) is verified (see [21]). So, applying Schur lemma again on (13) leads to the LMI (4) of lemma 1. □

2.2 Second formulation of the preliminary result

An alternative formulation of lemma 1 is given in a second lemma to ensure the stability and the $\mathcal{H}_\infty$ perfomance of system (1). This second lemma is used to prove theorem 1.

Lemma 2. *The uncertain system* (1) *is quadratically stable and satisfies the* $\mathcal{H}_\infty$ *performance* $J_{zw} < 0$ *for a given* $\gamma > 0$ *if there exist matrices* $P = P^T > 0$*,* G *and* $\nu > 0$ *such that*

$$\begin{bmatrix} AG + G^T A^T + \nu M M^T & P - G + G^T A^T + \nu M M^T & B & G^T C^T & G^T N_x^T \\ P - G^T + AG + \nu M M^T & -G - G^T + \nu M M^T & B & 0 & 0 \\ B^T & B^T & -\gamma I & D^T & N_w^T \\ CG & 0 & D & -\gamma I & 0 \\ N_x G & 0 & N_w & 0 & -\nu I \end{bmatrix} < 0. \quad (14)$$

■

Proof. Notice that from (4), by pre and post-multiplying it by bdiag $\left(G^{-1}, G^{-1}, 0, 0\right)$ and taking $P = G^{-1} P G^{-T}$, $G = G^{-T}$, we get

$$\begin{bmatrix} AG + G^T A^T & P - G + G^T A^T & B & G^T C^T \\ P - G^T + AG & -G - G^T & B & 0 \\ B^T & B^T & -\gamma I & D^T \\ CG & 0 & D & -\gamma I \end{bmatrix} + F_1 \Delta(t) F_2 + F_2^T \Delta^T(t) F_1^T < 0 \quad (15)$$

with $F_1 = \left[\, M^T \;\; M^T \;\; 0 \;\; 0 \,\right]^T$, $F_2 = [\, N_x G \;\; 0 \;\; N_w \;\; 0 \,]$.

The rest of the proof is similar to that of lemma 1. □

3 Robust observer-based controller design

In this part, we consider the following uncertain linear system

$$\dot{x} = (A + \Delta_A(t))\, x + (B_1 + \Delta_{B_1}(t))\, w + B_2 u \quad (16a)$$

$$z = C_1 x + D_{11} w + D_{12} u \quad (16b)$$

$$y = (C_2 + \Delta_{C_2}(t))\,x + (D_{21} + \Delta_{D_{21}}(t))\,w \tag{16c}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^q$ is the controlled output, $y(t) \in \mathbb{R}^p$ is the measured output, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^r$ is the disturbance vector. Without loss of generality, we assume that $m < n$.

The matrices $\Delta_A(t)$, $\Delta_{B_1}(t)$, $\Delta_{C_2}(t)$ and $\Delta_{D_{21}}(t)$ represent the parametric uncertainties and satisfy the following relation

$$\begin{bmatrix} \Delta_A(t) & \Delta_{B_1}(t) \\ \Delta_{C_2}(t) & \Delta_{D_{21}}(t) \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \end{bmatrix} \Delta(t) \begin{bmatrix} N_x & N_w \end{bmatrix} \tag{17}$$

with $\Delta^T(t)\Delta(t) \leqslant I_k$ and $\Delta(t) \in \mathbb{R}^{\ell \times k}$.

We aim to design an observer-based controller with the following structure

$$\dot{\eta} = H\eta + J_1 y + J_2 u \tag{18a}$$

$$u = \eta + Ey \tag{18b}$$

where $\eta(t) \in \mathbb{R}^m$ (with $m \leqslant n$) is the observer state and matrices H, J_1, J_2 and E are to be designed.

For this purpose, we first introduce the following definition.

Definition 1. *The uncertain system* (16) *is said to be robustly stabilizable based on function observer if there exist a gain matrix L, a function observer $\dot{\eta} = H\eta + J_1 y + J_2 u$ and a control law $u = \eta + Ey$ such that*

(i) $\lim_{t\to\infty} u(t) - Lx(t) = 0$ *if* $w(t) = 0$,

(ii) *the designed controller must be an observer-based one for the nominal system (i.e. a separation principle like condition is satisfied : the eigenvalues of $A + B_2L$ and H are those of the state matrix of the nominal closed-loop system),*

(iii) *the closed-loop system* (16)-(18) *is quadratically stable.* ■

Now, the problem to be treated can be stated as follows.

Problem 1. The objective is to establish a function observer (18a) and a control law (18b) such that

(i) $\lim_{t\to\infty} u(t) - Lx(t) = 0$ if $w(t) = 0$,

(ii) the designed controller must be an observer-based one for the nominal system (see item (ii) in definition 1).

(iii) the resulting closed-loop system (16)-(18) is quadratically stable and satisfies the $\mathcal{H}_\infty$ performance $J_{zw} < 0$ for a given $\gamma > 0$. ■

The approach used in this paper is to design the controller into two steps. The first one consists to use the item (i) of problem 1 i.e. to search for a state feedback gain L verifying $u(t) - Lx(t)$ for subsystem (16a)-(16b). Once this gain is found we then search in a second step, an observer-based controller (18) permitting to construct this gain L.

3.1 Synthesis of the robust state-feedback gain

In this part, we replace u by Lx in system (16a)-(16b) and get the following subsystem

$$\dot{x} = (A + B_2L + \Delta_A(t))\, x + (B_1 + \Delta_{B_1}(t))\, w \tag{19a}$$
$$z = (C_1 + D_{12}L)\, x + D_{11}w \tag{19b}$$

The above system is similar to system (1) and the result of section 2.2 can then be applied to build the gain L through the following theorem.

Theorem 1. *The uncertain system* (19) *is quadratically stable and satisfies the* $\mathcal{H}_\infty$ *performance* $J_{zw} < 0$ *for a given* $\gamma > 0$ *if there exist matrices* $P = P^T > 0$, G, Y *and* $\nu > 0$ *such that*

$$\begin{bmatrix} (1,1) & (1,2) & B_1 & G^TC_1^T + Y^TD_{12}^T & G^TN_x^T \\ (1,2)^T & -G-G^T+\nu M_xM_x^T & B_1 & 0 & 0 \\ B_1^T & B_1^T & -\gamma I & D_{11}^T & N_w^T \\ C_1G+D_{12}Y & 0 & D_{11} & -\gamma I & 0 \\ N_xG & 0 & N_w & 0 & -\nu I \end{bmatrix} < 0 \tag{20}$$

where

$$(1,1) = AG + G^TA^T + B_2Y + Y^TB_2^T + \nu M_xM_x^T$$
$$(1,2) = P - G + G^TA^T + Y^TB_2^T + \nu M_xM_x^T$$

Then the gain L *is given by*

$$L = YG^{-1}. \tag{21}$$

■

Proof. Note that G is invertible once the LMI (20) is satisfied, then the gain L can be computed.

By inserting $L = YG^{-1}$ in LMI (14), one obtains inequality (20) if A and C are replaced by $A + B_2L$ and $C_1 + D_{12}L$ respectively (*i.e.* compare system (19) with system (1)). Then this theorem is proved using lemma 2. □

3.2 Synthesis of the robust observer-based controller

Assume that the following constraint [15, 16]

$$E\begin{bmatrix} M_y & D_{21} \end{bmatrix} = 0 \tag{22}$$

holds. This assumption is justified in remark 2.

Using item (i) of definition 1, an observation error signal can be defined as

$$e = Lx - u = \Psi x - \eta - E(D_{21}w + M_y\Delta(t)(N_xx + N_ww)) \tag{23}$$

where

$$\Psi = L - EC_2. \tag{24}$$

Using (22), the observation error can be simplified as

$$e = \Psi x - \eta \tag{25}$$

and has the following dynamics

$$\begin{aligned}\dot{e} = He + (\Psi A - H\Psi - J_1 C_2)\, x + (\Psi \Delta_A(t) - J_1 \Delta_{C_2}(t))\, x \\ + (\Psi B_1 - J_1 D_{21})\, w + (\Psi \Delta_{B_1}(t) - J_1 \Delta_{D_{21}}(t))\, w + (\Psi B_2 - J_2)\, u\end{aligned} \tag{26}$$

and the unbiasedness of the nominal part corresponding to item (1) of definition 1 is achieved if and only if the closed-loop is quadratically stable and the two following conditions hold

$$0 = \Psi A - H\Psi - J_1 C_2, \tag{27a}$$

$$J_2 = \Psi B_2. \tag{27b}$$

If rank $L = q$ and using constraint (22), the nominal unbiasedness conditions (27a) is equivalent to [15, 16, 22]

$$\begin{bmatrix} E\ K \end{bmatrix} \overline{\Sigma} = \begin{bmatrix} 0\ 0\ L\overline{A} \end{bmatrix} \tag{28}$$

and the general solution of (28), if it exists, is given by

$$\begin{bmatrix} E\ K \end{bmatrix} = \begin{bmatrix} 0\ 0\ L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger + \overline{Z}(I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) \tag{29}$$

where

$$\overline{\Sigma} = \begin{bmatrix} M_y & D_{21} & C_2\overline{A} \\ 0 & 0 & \overline{C} \end{bmatrix} \tag{30}$$

$$\overline{A} = A(I - L^\dagger L), \tag{31}$$

$$\overline{C} = C_2(I - L^\dagger L), \tag{32}$$

$$K = J_1 - HE. \tag{33}$$

and $\overline{Z}$ is an arbitrary matrix of appropriate dimensions.

Remark 1. As in [15, 16, 22], the matrix L must be of full row rank. Notice that in case matrix L given by (21) is not full rank row, it suffices to add a small perturbation to fulfill this condition. ■

Relation (29) is the general solution of (28) if and only if [17]

$$\operatorname{rank} \begin{bmatrix} M_y & D_{21} & C_2\overline{A} \\ 0 & 0 & \overline{C} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 & L\overline{A} \\ M_y & D_{21} & C_2\overline{A} \\ 0 & 0 & \overline{C} \end{bmatrix} \tag{34}$$

or if and only if [15, 16, 22]

$$\operatorname{rank} \begin{bmatrix} M_y & D_{21} & LA \\ 0 & 0 & C_2 \\ 0 & 0 & L \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 & LA \\ M_y & D_{21} & LA \\ 0 & 0 & C_2 \\ 0 & 0 & L \end{bmatrix}. \tag{35}$$

Let $\xi^T(t) = \left[\ x^T(t) \quad e^T(t)\ \right]$ be the augmented closed-loop state vector. Then, if (35) holds, the closed-loop system (16)-(18) is given by

$$\dot{\xi} = \begin{bmatrix} A + B_2 L & -B_2 \\ 0 & H \end{bmatrix} !\xi + \underbrace{\begin{bmatrix} \Delta_A(t) & 0 \\ \Psi\Delta_A(t) - J_1\Delta_{C_2}(t) & 0 \end{bmatrix}}_{\Delta_{\mathbb{A}}(t)} \xi$$

$$+ \begin{bmatrix} B_1 \\ \Psi B_1 - J_1 D_{21} \end{bmatrix} w + \underbrace{\begin{bmatrix} \Delta_{B_1}(t) \\ \Psi\Delta_{B_1}(t) - J_1\Delta_{D_{21}}(t) \end{bmatrix}}_{\Delta_{\mathbb{B}}(t)} w \quad (36a)$$

$$z = \begin{bmatrix} C_1 + D_{12}L & -D_{12} \end{bmatrix} \xi + D_{11} w \quad (36b)$$

Due to relations (27a) and (29), the following relations are obtained

$$H = \widehat{A} - \overline{Z}\widehat{C} \quad (37)$$

$$\Psi B_1 - J_1 D_{21} = \widehat{B} - \overline{Z}\widehat{G} \quad (38)$$

where

$$\widehat{A} = LAL^\dagger - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \begin{bmatrix} C_2 A L^\dagger \\ C_2 L^\dagger \end{bmatrix}, \quad (39a)$$

$$\widehat{C} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) \begin{bmatrix} C_2 A L^\dagger \\ C_2 L^\dagger \end{bmatrix}, \quad (39b)$$

$$\widehat{B} = LB_1 - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \begin{bmatrix} C_2 B_1 \\ D_{21} \end{bmatrix}, \quad (39c)$$

$$\widehat{G} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) \begin{bmatrix} C_2 B_1 \\ D_{21} \end{bmatrix}. \quad (39d)$$

Remark 2. In system (36), the filtering error $e(t)$ is bilinear in the gain parameter $\overline{Z}$ due to the product $\overline{Z}\widehat{C}E$. This bilinearity is intrinsically linked to the unbiasedness condition (27a) . Indeed, the "bilinearity" $H\Psi$ in (27a) yields a gain K containing the product HE (see (33)). In order to avoid this kind of "bilinearity", we consider the constraint (22). ■

Notice that the uncertain terms combinations $\Psi\Delta_A(t) - J_1\Delta_{C_2}(t)$ and $\Psi\Delta_{B_1}(t) - J_1\Delta_{D_{21}}(t)$ are easily calculated by means of the system matrices. For example, using (24) and (29), the first term is given by

$$\Psi\Delta_A(t) - J_1\Delta_{C_2}(t) = \left(LM_x - \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} C_2 M_x \\ M_y \end{bmatrix} \right) \Delta(t) N_x$$

$$= \left(\left(LM_x - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \begin{bmatrix} C_2 M_x \\ M_y \end{bmatrix} \right) - \overline{Z} \left(\left(I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger \right) \begin{bmatrix} C_2 M_x \\ M_y \end{bmatrix} \right) \right) \Delta(t) N_x.$$

The matrix $\Psi\Delta_{B_1}(t) - J_1\Delta_{D_{21}}(t)$ is calculated in a similar way and the closed-loop system (36) becomes

$$\dot{\xi} = \left(\widehat{\mathbb{A}} + \Delta_{\mathbb{A}}(t) \right) \xi + \left(\widehat{\mathbb{B}} + \Delta_{\mathbb{B}}(t) \right) w \quad (40a)$$

$$z = \widehat{\mathbb{C}}\xi + D_{11}w. \tag{40b}$$

where

$$\begin{bmatrix}\Delta_{\mathbb{A}}(t) \ \Delta_{\mathbb{B}}(t)\end{bmatrix} = \mathbb{M}\Delta(t)\begin{bmatrix}\mathbb{N}_x \ \mathbb{N}_w\end{bmatrix} \tag{41}$$

with

$$\widehat{\mathbb{A}} = \begin{bmatrix} A + B_2L & -B_2 \\ 0 & \widehat{A} - \overline{Z}\widehat{C}\end{bmatrix}, \widehat{\mathbb{B}} = \begin{bmatrix} B_1 \\ \widehat{B} - \overline{Z}\widehat{G}\end{bmatrix}, \tag{42a}$$

$$\widehat{\mathbb{C}} = \begin{bmatrix} C_1 + D_{12}L \ -D_{12}\end{bmatrix}, \mathbb{M} = \begin{bmatrix} M_x \\ \widehat{M}_{y1} - \overline{Z}\widehat{M}_{y2}\end{bmatrix}, \tag{42b}$$

$$\widehat{M}_{y1} = LM_x - \begin{bmatrix}0 \ 0 \ L\overline{A}\end{bmatrix}\overline{\Sigma}^{\dagger}\begin{bmatrix} C_2M_x \\ M_y\end{bmatrix}, \tag{42c}$$

$$\widehat{M}_{y2} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^{\dagger})\begin{bmatrix} C_2M_x \\ M_y\end{bmatrix}, \tag{42d}$$

$$\mathbb{N}_x = \begin{bmatrix}N_x \ 0\end{bmatrix}, \mathbb{N}_w = N_w. \tag{42e}$$

Notice that the augmented system (36) is then rewritten in a similar form as system (1). It suffices to apply lemma 1 to get the unknown matrix $\overline{Z}$.

Theorem 2. *Assume that condition* (35) *holds. The robust $\mathcal{H}_\infty$ observer-based unbiased controller design problem 1 is solved under $E\,[\ M_y \quad D_{21}\ ] = 0$ and where L is given by* (21)*, if, for some $\mu > 0$, there exist $P_{11} = P_{11}^T > 0 \in \mathbb{R}^{n\times n}$, $P_{13} = P_{13}^T > 0 \in \mathbb{R}^{m\times m}$, $P_{12} \in \mathbb{R}^{n\times m}$, $G_1 \in \mathbb{R}^{n\times n}$, $G_3 \in \mathbb{R}^{m\times m}$ and $Y_3 \in \mathbb{R}^{m\times 2p}$ such that the two following LMI hold*

$$\begin{bmatrix} \Phi_{11} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \Phi_{21} & \Phi_{22} & \bullet & \bullet & \bullet & \bullet & \bullet \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \bullet & \bullet & \bullet & \bullet \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \bullet & \bullet & \bullet \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} & \bullet & \bullet \\ \Phi_{61} & \Phi_{62} & \Phi_{63} & \Phi_{64} & \Phi_{65} & \Phi_{66} & \bullet \\ \Phi_{71} & \Phi_{72} & \Phi_{73} & \Phi_{74} & \Phi_{75} & \Phi_{76} & \Phi_{77} \end{bmatrix} < 0 \tag{43}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{13}\end{bmatrix} > 0 \tag{44}$$

where "$\bullet$" is the transpose of the off-diagonal part and

$$\begin{aligned}
\Phi_{11} &= G_1A + G_1B_2L + A^TG_1^T + L^TB_2^TG_1^T + \mu N_x^TN_x \\
\Phi_{21} &= -B_2^TG_1^T \\
\Phi_{22} &= G_3\widehat{A} - Y_3\widehat{C} + \widehat{A}^TG_3^T - \widehat{C}^TY_3^T \\
\Phi_{31} &= P_{11} - G_1^T + G_1A + G_1B_2L \\
\Phi_{32} &= P_{12} - G_1B_2 \\
\Phi_{33} &= -G_1 - G_1^T \\
\Phi_{41} &= P_{12}^T
\end{aligned}$$

$$\begin{aligned}
\Phi_{42} &= P_{13} - G_3^T + G_3\widehat{A} - Y_3\widehat{C}\\
\Phi_{43} &= 0\\
\Phi_{44} &= -G_3 - G_3^T\\
\Phi_{51} &= B_1^T G_1^T + \mu N_w^T N_x\\
\Phi_{52} &= \widehat{B}^T G_3 - \widehat{G}^T Y_3^T\\
\Phi_{53} &= \Phi_{51}\\
\Phi_{54} &= \Phi_{52}\\
\Phi_{55} &= -\gamma I + \mu N_w^T N_w\\
\Phi_{61} &= C_1 + D_{12}L\\
\Phi_{62} &= -D_{12}\\
\Phi_{63} &= 0\\
\Phi_{64} &= 0\\
\Phi_{65} &= D_{11}\\
\Phi_{66} &= -\gamma I\\
\Phi_{71} &= M_x^T G_1^T\\
\Phi_{72} &= \widehat{M}_{y1}^T G_3^T - \widehat{M}_{y2}^T Y_3^T\\
\Phi_{73} &= \Phi_{71}\\
\Phi_{74} &= \Phi_{72}\\
\Phi_{75} &= 0\\
\Phi_{76} &= 0\\
\Phi_{77} &= -\mu I
\end{aligned}$$

The gain matrix $\overline{Z}$ *is then given by*

$$\overline{Z} = G_3^{-1} Y_3. \tag{45}$$

■

Proof. Under constraint (22), if the rank condition (35) holds, then the filter (18) is unbiased and system (36) represents the closed-loop given by the connexion of the uncertain system (16) with the controller (18) where matrices H, J_1, J_2 and E are given by (37), (33), (27b) and (29).

Note that G_3 is invertible once the LMI (43) is satisfied, then the gain $\overline{Z}$ can be computed.

Inserting $Y_3 = G_3\overline{Z}$ in the LMI (4), and taking $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{13} \end{bmatrix}$ and $G = \begin{bmatrix} G_1 & 0 \\ 0 & G_3 \end{bmatrix}$ yield inequality (43). Then using lemma 1, the robust quadratic stability and $\mathcal{H}_\infty$ performance required in item (ii) of problem 1 is ensured. Item (i) of problem 1 holds since $\lim_{t\to\infty} \xi(t) = 0$ if $w(t) = 0$ (and so $\lim_{t\to\infty} e(t) = 0$) if the closed-loop is robustly quadratically stable. Item (ii) is verified since the eigenvalues of $A + B_2L$ and H are those of the state matrix of the nominal part of closed-loop system (36). □

Remark 3. Notice that even if we are in the reduced-order case, we do not need to consider a block-diagonal matrix P (*i.e.* $P_{12} = 0$) as in [23] which can be conservative. This block-diagonal structure is then reported on matrix G. ■

Remark 4. Similarly to [24], it is easy to treat the multiobjective (and not mixed) control problem as it is possible to consider the same Lyapunov matrix P and different matrix G for each objective. ■

4 Numerical example

The different matrices of the uncertain system (16) are given by

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0.5 & 1 \\ 1 & -2 & -5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D_{21} = \begin{bmatrix} -0.5 & 0 \end{bmatrix}, M_x = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.3 & 0 & 0.1 \end{bmatrix}, M_y = \begin{bmatrix} 0 & 0.135 & 0 \end{bmatrix},$$

$$N_x = \begin{bmatrix} 0 & 0 & 0.135 \\ 0 & 0.135 & 0 \\ 0.135 & 0 & 0 \end{bmatrix}, N_w = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.135 \\ 0.135 & 0 \end{bmatrix}, \Delta(t) = \begin{bmatrix} a(t) & 0 & 0 \\ 0 & b(t) & 0 \\ 0 & 0 & c(t) \end{bmatrix}.$$

The nominal part of the uncertain system is unstable since A is non-Hurwitz.

The matrix $\Delta(t)$ is such that $a(t) = \cos(3t)$, $b(t) = 0.5 - 0.3\sin(2t)$ and $c(t) = 0.7\sin(4t)$.

We first apply theorem 1 and theorem 2 in a second time. The robust optimization gives $\gamma = 8.2$, $\nu = 8.3375$ and $\mu = 9.4793$.

Then the gain matrices $L = [\, L_1 \;\; L_2 \,]$ (21) and $Z = [\, Z_1 \;\; Z_2 \,]$ (45) are given by

$$L_1 = \begin{bmatrix} -0.2546 & -0.3333 \\ 1.1273 & -2.5091 \end{bmatrix}, L_2 = \begin{bmatrix} -0.3341 \\ -1.9966 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} -0.0845 & 0 \\ -1.0216 & 0 \end{bmatrix}, Z_2 = \begin{bmatrix} -0.0292 & -7.8692 \times 10^{-7} \\ -0.3516 & -5.1674 \times 10^{-6} \end{bmatrix}.$$

The filter-based controller matrices H, J_1, J_2 and E are given by (37), (33), (27b) and (29). This controller is of order 2.

The perturbations are given in figure 1. The numerical simulation is illustrated with figures 2 and 3.

Notice that, when the pertubation $w(t)$ vanishes, the state $x(t)$ and the estimation state error $e(t)$ converge to zero even if the uncertainties in $\Delta(t)$are not null.

5 Conclusion

In this paper, we have proposed an efficient method for robust reduced order observer based control for uncertain systems. Through the standard $\mathcal{H}_\infty$ techniques we first computed a robust state feedback gain and then used it to solve the nominal unbiasedness condition of the observer error dynamics. The observer matrices are then easily obtained by solving two LMI.

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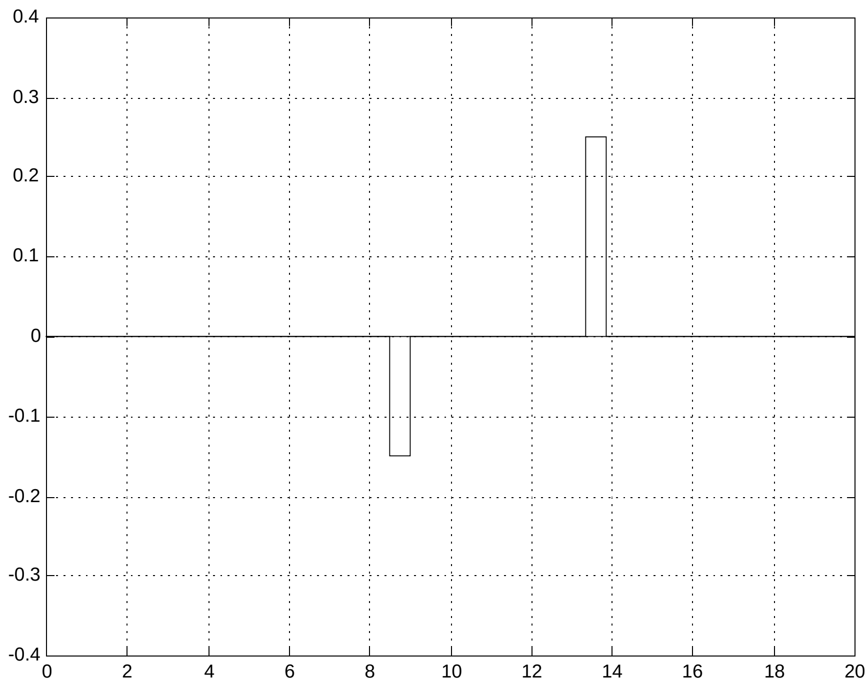

Fig. 1. Perturbation signal $w(t)$.

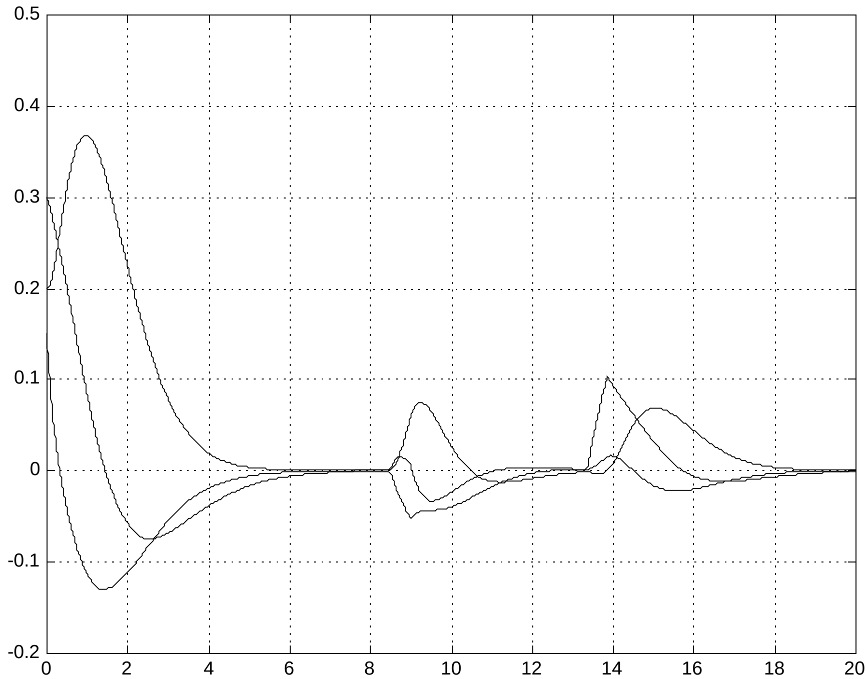

Fig. 2. Closed-loop states $x(t)$.

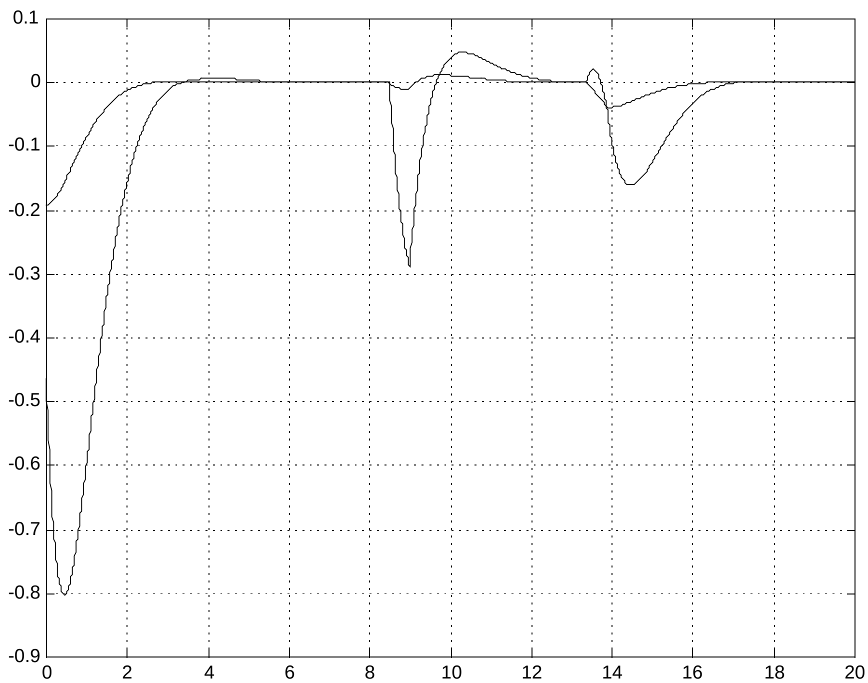

Fig. 3. Closed-loop error states $e(t)$.