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Diagnosis of uncertain linear systems: an interval approach

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Abstract. When modelling a physical process, there are always discreapencies between the modelled and real behaviours due to simplifications and neglected effects. Moreover, the parameters of the real process are never rigorously constant and may vary all along the time around a mean value. Thus the process state estimation will strongly depend on these variations which are unknown and thus may be considered as uncertainties. In order to obtain a guaranteed state reconstruction, the influence of these uncertainties have to be taken into account. An adapted representation allowing state reconstruction is the interval model approach which consists in providing a nominal model of the process influenced by uncertainties assumed to be represented by intervals with known bounds. The paper shows how state estimation may be obtained by using interval model approach; based on interval analysis, a set of state estimates is defined which is consistent with the available information (models, measurements, bounded uncertainties).

1 Introduction: state estimation for uncertain systems

The problem of estimating a system state using the exact knowledge of its input and output signals is well solved for processes with constant and known parameters (for example by using the well known Luenberger's observer or finite memory observer). However, real processes are often affected by disturbances and noises. Therefore, the design procedures of state observers have been extended to include the cases when disturbances and/or measurement noises are present.

When the system parameters are time varying and if their time dependency is unknown, the Luenberger's observer may also be applied. However the situation becomes more critical when the system under consideration is subjected to unknown disturbances or unknown inputs. When the systems are subjected to perturbations whose statistic characteristics are known, Kalman's filter may be

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used to reconstruct the system state. In fact the observer design techniques for processes subjected to uncertainties may be roughly divided into three groups.

The first group relies on robust estimation problem. The estimator is made robust in the face of both exogenous signals (unknown inputs for example) and model uncertainties. Within such framework, state estimation deals with the minimization of an induced norm of the transfer function from disturbances to estimation errors [27].

In the second group, the state estimation is performed on a reduced system corresponding to the unknown input free subsystem (which exists under some restrictive conditions). For that, the state equation is splitted into two parts, one being sensitive to the unknown input, the other being decoupled from this input. It is then possible, under specific conditions, to eliminate the unknown input influence on the state and the measurement equations by using an appropriate projection matrix. Thus, the observer is perfectly decoupled from the disturbances [13]. Another proposed observer design uses the sliding mode approach. Sliding observer is a high performance state estimator well adapted for nonlinear uncertain systems [29]. The sliding function of this observer is based on the estimation error of the available output of the system. Indeed, it consists in a classical Luenberger observer [17] to which is added a nonlinear term depending on the estimation error.

The last group of methods relies on the description of the uncertainties by known compact sets. The advantage of this description is the absence of hypothesis about the statistical properties of the uncertainties. The only need is the knowledge of the uncertainty bounds. In the field of diagnosis, robust model-based fault detection of dynamic systems using interval observers has been already addressed. One of the main techniques consists in checking whether the measurements of the output belong to the interval of all possible estimated outputs obtained considering uncertainty on model parameters [30]. In [23], the authors compare results of diagnosis obtained using interval models with those obtained with quantized systems describing the qualitative behaviour of a process. An original application in the field of flow rate sensor diagnosis is presented in [22] and the same idea is developed in [3] for data validation. It is important to point out that, although interval approaches need very little a priori information (only the uncertain parameter bounds), there are only a few published works on that subject, see for example [6] or [24].

In the present paper, after a brief overview of interval analysis, some results on guaranteed state estimation of linear systems with bounded uncertainties are summarized. Under the assumption that all uncertainties are bounded and belong to known sets, simple sets are built in order to have a very simple description of the state domain, such as orthotopes or parallelotopes guaranteed to contain the actual state vector. Fault diagnosis techniques for linear systems are derived from the estimation method.

An attractive way to tackle with uncertain parameters consists in considering bounded perturbations, the bounds of the later being a priori known. Indeed, as at each time instant the perturbations are unknown, it is impossible to determine the state of the system; thus, it seems reasonnable to estimate a domain in which the state lies. This problem is known as "set-valued state estimation" [16], [20], [32] and may be expressed under a general formulation when considering the following model of the system:

$$S \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) + v(k) \\ y(k) = Cx(k) + w(k) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^r$ the input, $y \in \mathbb{R}^m$ the observed output ; $v(k)$ and $w(k)$ are perturbations or bounded uncertainties, A, B and C are matrices of appropriate dimensions with possibly uncertain bounded parameters. Knowing the bounds $(v^- \quad v^+)$, $(w^- \quad w^+)$, $(A^- \quad A^+)$, $(B^- \quad B^+)$ and $(C^- \quad C^+)$ of the uncertainties, the model S, the input and output measurements, the aim is to estimate the bounds $(x^-(k) \quad x^+(k))$ of the state $x(k)$. At time k, the state domain is noted:

$$\mathcal{D}_{x,k} = \{x \;/\; x^-(k) \leq x(k) \leq x^+(k)\} \tag{2}$$

and, at time $k+1$, the new domain $\mathcal{D}_{x,k+1}$ is obtained using the previous domains $D_{x,k} \ldots D_{x,0}$, the new measurements $u(k)$, $y(k+1)$ and the bounds of the uncertainties. The case of linear systems is a priori simple, the state domain being represented with a polytope.

2 Interval analysis

This section reviews basic interval arithmetic operations for interval computations used in the paper. In the scope of the present paper, interval arithmetic makes it possible to take into consideration the parameter uncertainties and thus to provide strict bounds of the estimated variables. In the remaining, only real intervals are considered. As a definition, a real interval, denoted $[x]$, is a closed and connected subset of $\mathbb{R}$, defined by:

$$[x] = [x^- \quad x^+] = \{x \in \mathbb{R} \;/\; x^- \leq x \leq x^+\} \tag{3}$$

This definition can be extended to the v-dimension space: an interval vector $[x]$ of $\mathbb{R}^v$ is an v-dimensional rectangle or "box" of $\mathbb{R}^v$ and is the Cartesian product of intervals. The set of all boxes of $\mathbb{R}^v$ is denoted $\mathbb{IR}^v$.

The definition of real arithmetic operators and functions are extended to intervals, see some basic example in table 1. Given a function $f \;: \mathbb{R} \rightarrow \mathbb{R}$, a function $F \;:\; \mathbb{IR} \rightarrow \mathbb{IR}$ is an inclusion function of f if $\forall x \in [x], f(x) \in F([x])$. It is clear that this inclusion function is not unique; the natural inclusion function is obtained by substituting all real arguments and elementary functions (log, exp...) by their extension to intervals.

Definition or Operation	Formulation
Interval number	$x \in [x^- \;\; x^+]$, x^-: lower bound, x^+: upper bound $\mathbf{x} = [x^- \;\; x^+]$
Center Radius	$x_c = (x^- + x^+)/2$, $x_r = (x^+ - x^-)/2$ $\mathbf{x} = x_c + \mu x_r \quad \mid \mu \mid \leq 1$
Addition	$\mathbf{z} = \mathbf{x} + \mathbf{y} = [x^- + y^- \;\; x^+ + y^+]$ $\mathbf{z} = \mathbf{x} - \mathbf{y} = [x^- - y^+ \;\; x^+ - y^-]$
Multiplication by an interval	$\mathbf{z} = \mathbf{xy}$ $\mathbf{z} = [\min(a) \;\; \max(a)]$, $a = (x^-y^-, x^-y^+, x^+y^-, x^+y^+)$
Multiplication by a scalar	if $a > 0, \mathbf{z} = a\mathbf{x} = [ax^- \;\; ax^+]$ if $a < 0, \mathbf{z} = a\mathbf{x} = [ax^+ \;\; ax^-]$
Division	$\mathbf{z} = \mathbf{x} : \mathbf{y} = [x^- \;\; x^+][\frac{1}{y^+} \;\; \frac{1}{y^-}]$ unless $0 \in [y^- \;\; y^+]$ in which case the result of division is undefined

Table 1. Interval arithmetic operations

3 Ideal observer

In the paper, we consider a linear discrete time equation:

$$S \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) + v(k) \\ y(k) = Cx(k) \\ y_m(k) = y(k) + w(k) \end{cases} \tag{4}$$

where k is the time, $x \in \mathbb{IR}^n$ the state, $u \in \mathbb{R}^r$ the input, $y \in \mathbb{IR}^m$ the output, $y_m \in \mathbb{IR}^m$ the output measurement. The initial state $x(0)$ is asumed to belong to some prior compact set $\mathcal{D}_{x,0} \subset \mathbb{IR}^n$. The sequences $\{v(k)\}$ and $\{w(k)\}$ are unknown state and measurement noises also assumed to belong to known compact sets. More precisely, they are assumed to satisfy:

$$\begin{cases} v^- \leq v(k) \leq v^+ \\ w^- \leq w(k) \leq w^+ \end{cases}$$

where the indicated bounds are known at each time k.

The aim of the paper is to reconstruct the process state and does not concern the construction of the model and the identification of its parameters. However, let us just recall that identification algorithms of interval models provide for each parameter both lower and upper bound [10], [6]. A guaranteed state estimator, also named set-valued observer (SVO) constructs sets of admissible states which are consistent with the a priori bounds v^-, v^+, w^-, w^+ [28], [26]. In the linear case these sets generally can be described by polytopes.

The general idea for that construction consists in determining sets of possible states of the system which are consistent with the known bounds of the uncertainties, the model equation, and the current measurements (i.e. the *admissible* *domain*). The survey article of Milanese and Vicino [19] presents the basis statement of such methods. Particular applications in the field of bioprocesses are described in [5], [11], [31].

For example, consider the following system:

$$\begin{cases} x_1(k+1) = x_2(k), & x_1(0) = [0.878 \;\; 0.912] \\ x_2(k+1) = [0.7 \;\; 0.8]x_1(k) - 0.5x_2(k), & x_2(0) = [0.5 \;\; 0.6] \\ y(k) = x_1(k) + x_2(k) \end{cases} \tag{5}$$

The state at time 1 consistent with the initial state and the bounds of the perturbations is described by:

$$\begin{cases} 0.5 \quad \leq x_1(1) \leq 0.6 \\ 0.315 \leq x_2(1) \leq 0.48 \end{cases} \tag{6}$$

Consider now the measurement at time 1: $y(1) = [0.815 \; 0.95]$. The state consistent with this measurement has to verify:

$$0.815 \leq x_1(1) + x_2(1) \leq 0.95 \tag{7}$$

Figure 1 shows in the plane $\{x_2, x_1\}$ the set of the state values which are both consistent with the state equation and the measurement. This set (in grey color on the figure) results from the intersection of the zonotope defined by equation (6) and the strip defined by (7).

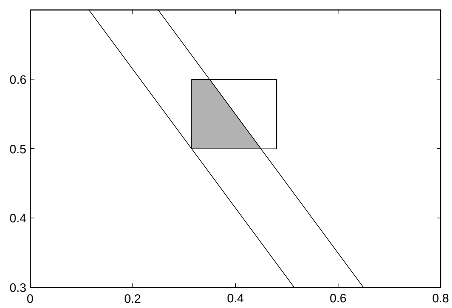

Fig. 1. Set of admissible state values at time 1

3.1 Ideal observer structure

We are now interested in constructing the admissible domain, denoted $\mathcal{D}_{x,k}$ which is consistent with the current measurement, the model of the system and the bounds of the uncertainties. The initial state $x(0)$ of system (4) is assumed to belong to the compact set $\mathcal{D}_{x,0} \subset \mathrm{I\!R}^n$. The uncertainties v and w are bounded and, without severe restriction of the approach, they are supposed constant all along the time:

$$| v(k) | \leq \delta_v \quad | w(k) | \leq \delta_w$$

Thus at time k, $\mathcal{D}_{x,k}$ represents the set of all the state values, $x(k)$, consistent with the available information I_k:

$$I_k = \{\mathcal{D}_{x,0},\ \{u(i-1), y_m(i), | v(i) | \leq \delta_v, | w(i) | \leq \delta_w\}_{i=1}^{k}\}$$

- **Observation step.** At time k, the observation allows to deduce the possible values taken by the state $x(k)$ considering the available measurement $y_m(k)$. First, the sets of the values of the system outputs which are consistent with the measurements $y_m(k)$ and the uncertainty bounds are defined by:

$$\mathcal{D}_{y,k} = \{y \ / \ \ | y - y_m(k) | \leq \delta_w\}$$

Then the set of the possible state values $x(k)$ is deduced:

$$\mathcal{D}_{x,k}^{y} = \{x \ \in \mathbb{R}^n / \ Cx \in \mathcal{D}_{y,k}\} \tag{8}$$

More simply, the domain $\mathcal{D}_{x,k}^{y}$ represents the set of possible states at time k based on the single measurement $y_m(k)$ only and is obtained through a set-inversion technique. This problem may be solved analytically for "academic" examples where the structure of the function Cx allows the extraction of the state x. In other situations, numerical algorihms are used to approximate the true domain; for that purpose, see for example the package SIVIA [14].
- **Prediction step.** The prediction step involves the evaluation of the state equation to propagate the current set of states and parameters until the next measured value is available. Thus we define the domain:

$$\mathcal{D}_{x,k}^{+} = \{Ax + Bu(k) + v \ / \ x \in \mathcal{D}_{x,k},\ | v | \leq \delta_v\} \tag{9}$$

- **Correction step.** Finally, the admissible domain $\mathcal{D}_{x,k}$ compatible with the information I_{k+1} is obtained by the intersection of the two domains obtained using the model equation and the measurement equation, as follows:

$$\mathcal{D}_{x,k+1} = \mathcal{D}_{x,k}^{+} \cap \mathcal{D}_{x,k+1}^{y} \tag{10}$$

Remark 1. In some situation, depending on the measurement values, it is possible to obtain an empty intersection when using (10) because of contradictory knowledges. If necessary, this problem may be overcome by using an expansion operator allowing to enlarge the domain of the admissible output [12]. However, when the objective of the state estimation is included in the general framework of diagnosis, this particular situation may be analysed in order to detect and identify faults. Indeed, an empty set is a good way to detect the presence of outliers or more generally a dysfunction of the system.

Remark 2. Repeated application of the intersection procedure (10) generally leads to a complex shape (and consequently a complex description) of the set

$\mathcal{D}_{x,k}$. Both computation time of the intersection and the storage memory necessary to describe this shape considerably increase and may be serious drawbacks for real-time application. To overcome this problem, it is then possible to reduce the complexity of the procedure by evaluating an approximated (but guaranteed) set $\hat{\mathcal{D}}_{x,k}$ of $\mathcal{D}_{x,k}$. This has been suggested in several papers, in particular [7], [18]. In the paper [9], the authors suggest an algorithm to limit the order of zonotopes obtained through successive integrations of the state equation, the idea is to over-approximate a zonotope by a zonotope of lower order. Consequently, the SVO algorithm is structured as follows:

- Step 0 Initialize $\mathcal{D}^+_{x,0}$ with $\mathcal{D}_0$. Let $k = 1$
- Step 1 Collect the data $u(k)$ and $y_m(k)$
- Step 2 Characterize the output domain $\mathcal{D}_{y,k} = \{y \ / \ \mid y - y_m(k) \mid \leq \delta_w\}$
- Step 3 Characterize the state domain $\mathcal{D}^y_{x,k} = \{x \ \in \mathbb{R}^n / \ Cx \in \mathcal{D}_{y,k}\}$
- Step 4 Characterise the admissible state domain $\mathcal{D}_{x,k} = \mathcal{D}^+_{x,k-1} \cap \mathcal{D}^y_{x,k}$
- Step 5 Reduce the domain complexity $\hat{\mathcal{D}}_{x,k} \supseteq \mathcal{D}_{x,k}$
- Step 6 Predict the state set $\mathcal{D}^+_{x,k} = \{Ax + Bu(k) + v \ / \ x \in \hat{\mathcal{D}}_{x,k}, \ \mid v \mid \leq \delta_v\}$
- Step 7 Increase $k := k + 1$, go to Step 1

3.2 Example

For example, let us consider the linear system:

$$\begin{array}{ll} x(k+1) = [0.50 \quad 0.65]x(k) + 0.25, & x(0) \in [0.10 \quad 0.20] \\ y(k) = 2x(k) + w(k), & \mid w(k) \mid \leq 0.08 \end{array} \tag{11}$$

This system is sensitive to three uncertainties, which respectively affect its initial state, the output measurements and one parameter of the system.

From the initial state, we have the a priori domain $\mathcal{D}^+_{x,0} = [0.10 \quad 0.20]$. Considering the first measure $y_m(1) = 0.44$, the output domain is given by $\mathcal{D}_{y,1} = [0.36 \ \ 0.52]$, then the state domain is obtained: $\mathcal{D}^y_{x,1} = [0.18 \ \ 0.26]$. By intersecting the two state domains $\mathcal{D}^+_{x,0}$ and $\mathcal{D}^y_{x,1}$, the admissible state domain is derived:

$$\mathcal{D}_{x,1} = [0.18 \quad 0.20]$$

For this particular example, there is no need to reduce the domain complexity. The predicted state domain is directly evaluated using the state equation:

$$\mathcal{D}^+_{x,1} = [0.34 \quad 0.38]$$

This elementary computational cycle may be restarted, using all the available measurements and the current admissible state domain.

4 Worst case estimation

Uncertain systems described by (4) are characterized by time varying parameters. Even if the value of the parameters are not precisely known it is realistic to assume that their range of variation are a priori known or identified. Knowing the bounds of the parameter values, the state estimation must be carried out for all possible values of the parameters at each time.

Let study the worst case provoked by the uncertainties. The vocable *worst* is understood in the sense that it results in the largest interval for the state estimate. It appears that the worst sequence of parameter values on a time window, is not always the sequence of the worst parameter values at each time in the window. Thus all possible time evolutions of the parameter values must be envisaged to guarantee that x lies in $[x^- \ x^+]$.

4.1 A closed-form solution

The proposed method consists in searching a closed-form expression of the state $x(k)$, depending only on the initial state $x(0)$ and the sequence of the uncertainties $\{\eta(0), \eta(1), \ldots, \eta(n-1)\}$. At each time, the study of this closed-form expression for all possible values of the uncertainties should allow to obtain an interval estimate for each component of the state vector. Let us consider a linear system of the form:

$$\begin{cases} x(k) = A(\eta(k-1))x(k-1) + Bu(k-1) \quad x(0) \in \mathcal{D}_{x,0} \\ y(k) = Cx(k) \end{cases} \tag{12}$$

where $\eta(k)$ denotes the uncertainties. With no loss of generality, it is assumed that the uncertainties are normalized: $|\eta(k)| \leq 1$, where the operator $\leq$ applied to a vector means that each component of $\eta(k)$ is bounded by 1, the state vector can be written as:

$$x(k) = f(x(0), u(k-1), \ldots, \ u(0), \eta(k-1), \ \ldots, \ \eta(0)) \tag{13}$$

Defining the vector η_k and u_k obtained by concatenating the values of the uncertainties and the inputs on the time window $[0, \ k-1]$:

$$\begin{aligned} \eta_k &= \big(\eta^T(0) \ldots \eta^T(k-1)\big) \\ u_k &= \big(u^T(0) \ldots u^T(k-1)\big) \end{aligned} \tag{14}$$

the state vector can be written as:

$$x(k) = f(x(0), u_k, \eta_k) \tag{15}$$

Then, for a given time k, the set $\mathcal{D}_{x,k}$ in which lies the state vector is given by:

$$\mathcal{D}_{x,k} = \{x \in \mathbb{R}^n \ / \ x = f(x(0), u_k, \eta_k), \quad x(0) \in \mathcal{D}_{x,0}, | \ \eta_k \ | \leq 1\} \tag{16}$$

As previously argued, this general formulation leads to tedious calculus. Moreover, it may result in very complex shape of the state domain $\mathcal{D}_{x,k}$. Regardless to a possible over-estimation, the state domain can be bounded by its interval hull $\Box S_{x,k}$, the latter being easier to evaluate than the former. For instance, $\Box S_{x,k}$ can be defined by:

$$\Box S_{x,k} = [x^-(k) \quad x^+(k)] \tag{17}$$

The upper and lower bounds of the state estimate are given by (18), where the operators min and max are applied to each component of $x(k)$.

$$\begin{cases} x^+(k) = \max\limits_{\eta_k, |\eta_k| \leq 1} x(k) \\ x^-(k) = \min\limits_{\eta_k, |\eta_k| \leq 1} x(k) \end{cases} \tag{18}$$

This model-based approach provides state estimates taking into consideration the worst sequence of the uncertainties. The only conservatism introduced by this method is due to the fact that optimization procedures to obtain the bounds of each component of $x(k)$ are run independently. It does not take into account that the components of $x(k)$ are coupled by the uncertainties which simultaneously influence the different components of $x(k)$. This conservatism may result in a larger estimated state domain, containing the real state domain.

4.2 A sub-optimal solution

Despite its formal simplicity, the computing load of the previous proposed method increases with k, because of the increasing number of uncertainties. Unfortunately, there is no link between $\mathcal{D}_{x,k-1}$ and $\mathcal{D}_{x,k}$, defined in (16), since all possible sequences of uncertainties must be envisaged for each case and the worst η_k is not always composed of $(\eta_{k-1}, \eta^T(k))$. A solution to limit the computation complexity and to allow on-line estimation, is the estimation on a sliding window of width $\ell + 1$. The value of ℓ depends on the system dynamics, and may be chosen to be equal to two or three times the (pseudo) natural period. With a constant width window, the dimension of η_k is constant. The procedure becomes the following: the state vector at time $k + \ell$ depends on the first value of the state vector in the window and the inputs:

$$\begin{aligned} x(k+\ell) &= f_\ell(x(k), u_k^\ell, \eta_k^\ell) \\ u_k^\ell &= (u^T(k) \dots u^T(k+\ell-1)) \\ \eta_k^\ell &= (\eta^T(k) \dots \eta^T(k+\ell-1)) \end{aligned} \tag{19}$$

The consequent state domain is similar to the one defined in (16), except that the number of optimization parameters is constant when the time window is sliding.

A simpler, but more conservative, solution consists in using an *almost time invariant approach*. For that purpose, the uncertainties are assumed to be constant (and equal to η) on the time window

$$x(k+\ell) = f_\ell(x(k), u(k+\ell-1),\ \ldots,\ u(k), \eta) \tag{20}$$

In this case, the number of uncertain parameters is constant and does not depend on this width. The main remaining question is the determination of the window width, some answers can be found in [21].

5 Application to diagnosis

The previous sections were dedicated to the design of an interval state observer. The interval estimate is obtained by the intersection between the prediction issued from the state equation and the state deduced from the observation equation of the outputs of the system. When applying this procedure, one assumes implicitly that both sources of information are coherent and are not affected by any biases. In this case, the resulting estimate is representative of the actual state of the system. For diagnosis purpose, it is precisely the problem of inconsistency of information which prevails. The impossibility of merging the two sources of information (case where the two domains $\mathcal{D}^+_{x,k-1}$ and $\mathcal{D}^y_{x,k}$ are disjoined) reveals the presence of measurement errors or, more exactly the incompatibility between the measurements and the model of the system. During this diagnosis analysis, one is particularly interested in estimating the system outputs, while the state estimation is only an intermediate step. Indeed, the output estimate could be compared with the measured outputs in order to generate the so-called residuals, whereas this analysis is not possible for the system state. This comparison between the estimated outputs and the measured ones will be made in an interval framework in order to take into account of the bounded uncertainties affecting both the measurements and the model of the system.

5.1 Residual generation

The output domain $\mathcal{D}^+_{y,k}$ predicted by the model of the system is deduced from the state domain $\mathcal{D}^+_{x,k}$:

$$\begin{aligned}\mathcal{D}^+_{y,k} = \{y^+ \ / \ y^+ = Cx^+ + w,\ x^+ = Ax + Bu(k) + v,\\ x \in \mathcal{D}_{x,k},\ \mid v \mid \leq \delta_v,\ \mid w \mid \leq \delta_w\}\end{aligned} \tag{21}$$

In the same way, the admissible output domain $\mathcal{D}_{y,k+1}$, evaluated from the measurements, is defined by:

$$\mathcal{D}_{y,k+1} = \{y \ / \ \mid y - y_m(k+1) \mid \leq \delta_w\} \tag{22}$$

Consequently, starting from these two domains, an interval residual can be defined in the following way:

$$r_{k+1} = \mathcal{D}^+_{y,k} \cap \mathcal{D}_{y,k+1} \tag{23}$$

A fault is detected if $r_{k+1} = \emptyset$. One should note that, determining the frontiers of the two domains at every time may result in an important computational load. For this reason, the exact domain is often approximated by a domain of simpler form, for example presenting less vertice. Thus, modifying (23), a fault is detected if the following residual is empty:

$$\hat{r}_{k+1} = \hat{\mathcal{D}}^+_{y,k} \cap \mathcal{D}_{y,k+1} \tag{24}$$

where $\hat{\mathcal{D}}^+_{y,k}$ is an overestimation of $\mathcal{D}^+_{y,k}$, i.e.:

$$\mathcal{D}^+_{y,k} \subseteq \hat{\mathcal{D}}^+_{y,k} \tag{25}$$

If $\hat{\mathcal{D}}^+_{y,k}$ is easier to compute than $\mathcal{D}^+_{y,k}$, the residuals (24) are simpler to compute that (23); however, the domain $\hat{\mathcal{D}}^+_{y,k}$ results to detect less faults than the domain $\mathcal{D}^+_{y,k}$. In fact, if $\hat{\mathcal{D}}^+_{y,k} \cap \mathcal{D}_{y,k+1} \neq \emptyset$ and $\mathcal{D}^+_{y,k} \cap \mathcal{D}_{y,k+1} = \emptyset$, then a fault occurred but was not detected [4]. Thus the difficulty is to define a compromise between the complexity of the determination of the state domain or the output domain and the tolerable rate of no detection. The reader will notice that, compared to what was presented at the section 3, within the framework of the state estimation, the reduction of complexity was not carried out on the same domain. According to the difficulty of implementation, the user can choose to do this reduction at any step of the proposed algorithms, keeping in mind that this latter always generates an approximation.

The preceding formalism makes it possible to detect inconsistencies of data. Nevertheless, this diagnosis remains a little vague, thus it is worth specifying how, in a more general way, to highlight the occurrence of a fault. A solution consists in computing the interval state estimate using only a part of the output measurements. Analogously to the design of banks of dedicated observers in [8], p domains can be built, where each domain is computed with only one component of the measurement vector y_m.

In the following, a procedure of change detection of the operating mode of a system is proposed knowing, at every moment, its inputs and outputs.

5.2 Change detection of operating mode

In the framework of supervised diagnosis, one admits that all the failures affecting a system are known. Each failure results in a given and known operating mode: each normal operating mode or dysfunctioning mode is thus described by a model. Therefore, the failure i is associated to the particular model:

$$M_i \begin{cases} x(k+1) = A_i x(k) + B_i u(k) + v(k) \\ y_m(k) = C_i x(k) + w(k) \end{cases} \tag{26}$$

where v and w already denotes the uncertainties affecting the model and the measurement system.

In a more general way, the set of the models $M_i,\ i = 1 \ldots N$ represents all the operating modes including the healthy modes related to the absence of faults. Thus, the diagnosis consists, starting from available measurements, in determining which model, among a set of models with uncertain parameters, is compatible with the measurements and the bounds of the uncertainties. The selected principle is the invalidation of model. At one moment k, each model M_i allows to predict the state x in an interval form (domain $\mathcal{D}^+_{x,k,i}$). If a prediction is incompatible with the state $\mathcal{D}^y_{x,k+1,i}$ deduced from the measurements y, then the corresponding model does not reflect the current situation and thus the system does not operate in the corresponding mode.

The algorithm to be implemented, inspired of [18], is then the following:

- Step 0. Define an initial state domain $\mathcal{D}^+_{x,0}$, $k = 1$.
- Step 1. Collect the data $u(k)$ and $y_m(k)$
- Step 2. Characterize the output domain $\mathcal{D}_{y,k}$

$$\mathcal{D}_{y,k} = \{y \ / \ \mid y - y_m(k) \mid \leq \delta_w\}$$

- Step 3. Characterize the admissible state domain starting from the output domain, for $i = 1, \ldots, N$:

$$\mathcal{D}^y_{x,k,i} = \{x \in \mathbb{R}^n \ / \ C_i x \in \mathcal{D}_{y,k}\}$$

- Step 4. Characterize the admissible state domains $\mathcal{D}_{x,k,i}$, for $i = 1, \ldots, N$:

$$\mathcal{D}_{x,k,i} = \mathcal{D}^+_{x,k-1,i} \cap \mathcal{D}^y_{x,k,i}$$

- Step 5. Reduce the domain complexity, for $i = 1, \ldots, N$:

$$\mathcal{D}^+_{x,k,i} \subseteq \hat{\mathcal{D}}^+_{x,k,i}$$

- Step 6. Characterize the admissible domain using prediction based on the i^{th} model, for $i = 1, \ldots, N$:

$$\mathcal{D}^+_{x,k,i} = \{A_i x + B_i u(k) + v \ / \ x \in \hat{\mathcal{D}}_{x,k,i}, \ \mid v \mid \leq \delta_v\}$$

- Step 7. Increase $k = k + 1$ and go to Step 1.

The interpretation of the various domains $\mathcal{D}_{x,k,i}$ is done in the following way. Let us recall that $\mathcal{D}_{x,k,i},\ i = 1 \ldots N$ represents the set of all the admissible states consistent with the available measurements and the uncertainty bounds, considering the i^{th} model. If the particular domain $\mathcal{D}_{x,k,i_0}$ is empty, it means that the current evolution is not correctly described by the i_0^{th} operating mode.

Obviously, if two domains $\mathcal{D}_{x,k,i_1}$ and $\mathcal{D}_{x,k,i_2}$ are not simultaneously empty, there is an ambiguity. Indeed, the two modes i_1 et i_2 are then candidates to describe the corresponding situation. In this case, additional information is necessary to refine the diagnosis and to distinguish the modes i_1 and i_2. The concept of persistence can be a recourse useful to this discrimination. The method is to build and analyze the various domains at consecutive moments, the vacuity of the domains is then analyzed over one more significant duration.

This situation of ambiguity would deserve many other developments on the analysis of its origins. An important point to analyze would be the separability or the discernability of the operating modes which are related to the distances between corresponding models (the concept of distance remaining to be clearly defined) and to the influence of the noise of measurements.

6 Example: search for active mode

Let us consider a system which can be in one of the three following configurations: normal mode of operation ($i = 0$), first abnormal mode of dysfunction ($i = 1$), second abnormal mode of dysfunction ($i = 2$). It is assumed that the models corresponding to these three modes are known, as well as the measurements of the inputs and outputs collected on the system. The problem arising is to determine the current operating mode of the system at every moment.

6.1 System models

To simplify, the three models are taken as relations between the outputs $y_i = (y_{i1} \quad y_{i2})^T$ and the inputs $x = (x_{i1} \quad x_{i2})^T$ of the system:

$$M_i \begin{cases} y(k) = X(k)\theta_i(k) \\ \theta_i(k) = \theta_{0,i} + T_i\eta_i(k), \quad \mid \eta_i(k) \mid \leq 1 \end{cases} \tag{27}$$

For each mode i, the uncertain parameter $\theta_i(k)$ is described by its nominal value $\theta_{0,i}$, and uncertainty modeled by a normalized variable $\eta_i(k)$ and its distribution matrix T_i. In the example, the uncertain parameters are defined by:

$$\begin{cases} \theta_0(k) = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix} + \begin{pmatrix} 0.1 & 0.2 & -0.2 \\ 0.3 & 0.1 & 0.2 \end{pmatrix} \eta_0(k) \\ \theta_1(k) = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix} + \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{pmatrix} \eta_1(k) \\ \theta_2(k) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0.1 & -0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 \end{pmatrix} \eta_2(k) \end{cases} \tag{28}$$

The reader will notice that this application example is slightly different from the previously described case. Here, the considered models are described by input-output relations rather than state space models. It results that there is no need to characterize any admissible state domains but only admissible output domains. This is the reason why the used notation are slightly modified. In particular, $\mathcal{D}_{y,k,i}$ will denote the admissible output domains for each model.

6.2 Improving the output estimation

Following the previously stated principle, the outputs $y(k)$ can be predicted from the inputs $x(k)$ of each of the three operating models:

$$\mathcal{D}_{y,k,i} = \{y \ / \ y = X(k)(\theta_{0i} + T_i\eta_i(k))\}, \mid \eta_i(k) \mid \leq \ 1 \tag{29}$$

The procedure is illustrated by the figures 2 and 3 obtained with:

$$X(k) = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

Let us consider the model M_1, at an unspecified instant for which the two components of the uncertain parameter vector are given by:

$$M_1 \begin{cases} \theta_{11} = 3.5 + 0.1\eta_{1,1} + 0.2\eta_{1,2} \\ \theta_{12} = 4 + 0.3\eta_{1,1} + 0.1\eta_{1,2} \end{cases} \tag{30}$$

It should be noticed that the two components of θ_1 are coupled via two standardized uncertainties $\eta_{1,1}$ and $\eta_{1,2}$. The couplings are highlighted by eliminating $\eta_{1,1}$ or $\eta_{1,2}$ in (30), which can be rewritten as:

$$\begin{cases} 3\theta_{11} - \theta_{12} = 6.5 + 0.5\eta_{1,2} \\ \theta_{11} - 2\theta_{12} = -4.5 - 0.5\eta_{1,1} \end{cases} \tag{31}$$

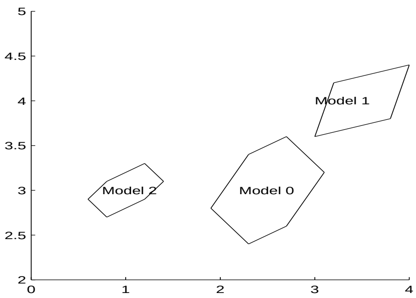

Fig. 2. Parameter domain.

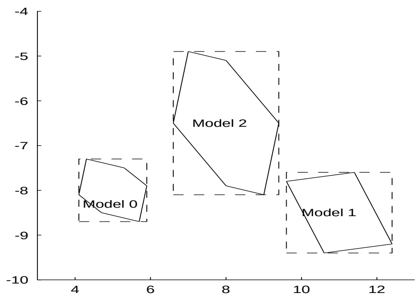

Fig. 3. Output domain.

Thus, taking into account the bounds of $\eta_{1,1}$ and $\eta_{1,2}$, the domain defining the components θ_{11} and θ_{12} is defined by the polytope:

$$\begin{cases} 3.2 \leq \quad \theta_{11} \quad \leq 3.8 \\ 3.6 \leq \quad \theta_{12} \quad \leq 4.4 \\ 6 \leq 3\theta_{11} - \theta_{12} \leq 7 \\ -5 \leq \theta_{11} - 2\theta_{12} \leq -4 \end{cases} \tag{32}$$

The domain corresponding to this description is represented on the figure 2. Same construction applies to the polytopes resulting from the two other models

M_0 and M_2. Then the domain $\mathcal{D}_{y,k,i}$ may be obtained. For example, $\mathcal{D}_{y,k,1}$, using (29) and (30), may be constructed using the components of y:

$$\begin{cases} y_1 = 11 + 0.5\eta_{1,1} + 0.5\eta_{1,2} \\ y_2 = -8.5 - 0.8\eta_{1,1} - 0.1\eta_{1,2} \end{cases} \tag{33}$$

Taking into account the coupling between the two components and the values of the bounds of $\eta_{1,1}$ and $\eta_{1,2}$, the following inequalities define the domain $\mathcal{D}_{y,k,1}$:

$$\begin{cases} 10 \leq & y_1 & \leq 12 \\ -9.4 \leq & y_2 & \leq -7.6 \\ 4.1 \leq & 0.8y_1 + 0.5y_2 & \leq 4.8 \\ -35 \leq & y_1 + 5y_2 & \leq -28 \end{cases} \tag{34}$$

The domain described by (34) is represented on the figure 3 in the plane $\{y_1, y_2\}$ of the output. At a given time instant, the parameter θ belongs to one of the 3 polytopes of figure 2 and the measurement y belongs to one of the 3 polytopes of figure 3. The "complex" shape of these domains result from the coupling between the two outputs (see for example (34)). In the sequel, in order to simplify the fault detection procedure, the selected domain is the smallest zonotope $\square_{y,k,i}$ containing the exact domain $\mathcal{D}_{y,k,i}$. For example, the domain $\mathcal{D}_{y,k,1}$ defined by (33) is approximated by the zonotope $\square_{y,k,1}$:

$$\begin{cases} 10 \leq y_1 \leq 12 \\ -9.4 \leq y_2 \leq -7.6 \end{cases} \tag{35}$$

The figure 3 compares the zonotope $\square_{y,k,i}$ (dashed lines) with the corresponding polytope $\mathcal{D}_{y,k,i}$. For this particular example, one can note that the three zonotopes $\square_{y,k,i}, i = 1, 2, 3$ are disjoined.

The definition of these zonotopes must be done on-line at each instant. A more exploitable information consists in using an interval representation of each component $y_{ij}(k)$ of the output predicted by each model.

6.3 Generation of the active mode indicators

The figure 4 shows the bounds of the two outputs y_{i1}, y_{i2} (in columns) for each model (in rows). The simulation was done on the horizon $[0 \quad 30]$, the changes of operating mode occured at the moments 12 (switching from M_0 to M_1) and 20 (switching from M_1 to M_2). On each figure, the output $y(k)$ is drawn in dash-line in order to be compared with its estimates based on each model. Therefore the admissible output domains are defined for the three models. The active mode is determined by analysing the measured output of the system together with these three domains. In the considered example, the output is not corrupted by any bounded noise and can thus be directly compared with the bounds of the output interval of the three models. For each component of the output vector and each model M_i, one defines the following residuals:

$$r_{ij}(k) = [y_{ij}^-(k) - y_j(k),\ y_{ij}^+(k) - y_j(k)], \quad i = 0, 1, 2 \quad j = 1, 2 \tag{36}$$

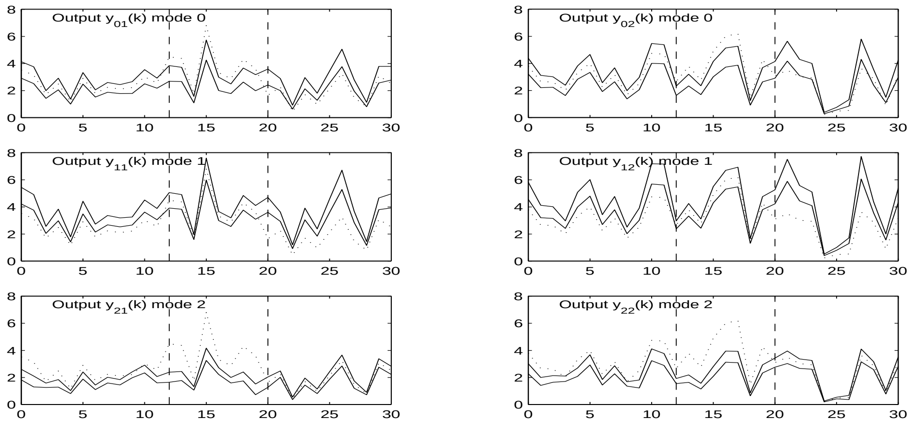

Fig. 4. Interval type outputs estimated by the three models.

y_{ij}^- and y_{ij}^+ being the bounds of the model M_i output and y_j, the jth component of the observed output. The residuals are depicted on figure 5. Clearly, the active mode can be detected when the interval residual contains the value 0. The detector able to carry out the analysis of the residuals is based on the sign of the residual bounds:

$$\tau_{ij}(k) = \frac{1}{2}\left(1 - \mathrm{sign}(y_{ij}^-(k) - y_j(k))(y_{ij}^+(k) - y_j(k))\right) \tag{37}$$

where it is recalled that the indexes i and j respectively relates to the i^{th} model and the j^{th} component of the output. Their evolutions are represented on the figure 6. One can give the following interpretation: a value of τ equal to 1 (respectively 0) testifies to (respectively invalidates) the membership of the origin to the interval residual. The result presented on this figure is discriminating with respect to the changes of mode. The examination of the indicators τ_{01} and τ_{02} built from the model corresponding to mode 0, makes it possible to state the following results: between the moments 0 and 12, the mode 0 is active, between the moments 12 and 30, three false detections appear (they can be easily eliminated by a nonpersistence test). The analysis of the residuals resulting from the two other models confirms and supplements this conclusion. Globally, these six graphs are coherent and contribute to well defining the active mode, at every moment.

7 Conclusion

Undoubtedly, taking benefits of any knowledge about uncertainties is one of the fundamental points of current research and development in system analysis. This communication was focused on the bounded approach which consists in representing each uncertainty by an interval. The propagation of these intervals along the time in the system equation then results in defining observers of the interval type, which themselves, provide estimates of the interval type of the state of the system. Within the framework of the diagnosis, that leads to define fault indicators of the interval type. This study offers many prospects. Among them, in order to be efficient, the different bounds of the state and measurement noises must be evaluated. These bounds can be determined a priori from the "quality" of the model and that of the measurements. They can also be estimated from experimental input-output data assuming they are not corrupted by faults. Some attempts in that way were already published [2], [1]. Moreover, the decision logic elaborated on the basis of the interval residuals can also be enhanced. The fusion of the results delivered by the indicators of mode change remains to be made and, in particular, the use of the exoneration principle could be useful for providing a coherent diagnosis.

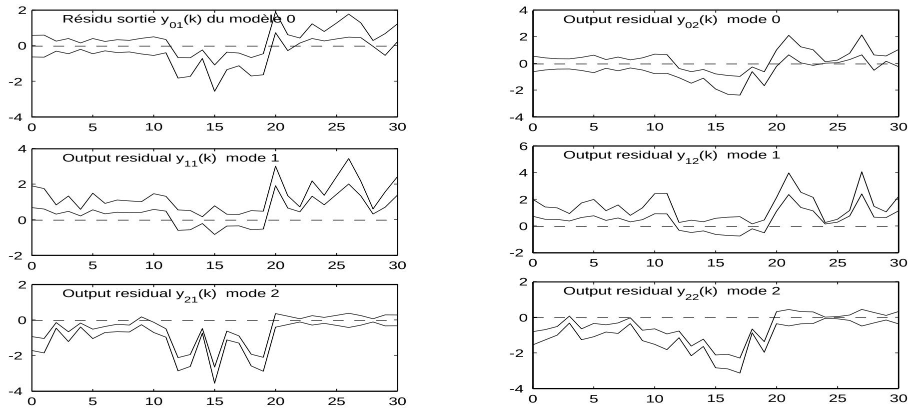

Fig. 5. Residuals

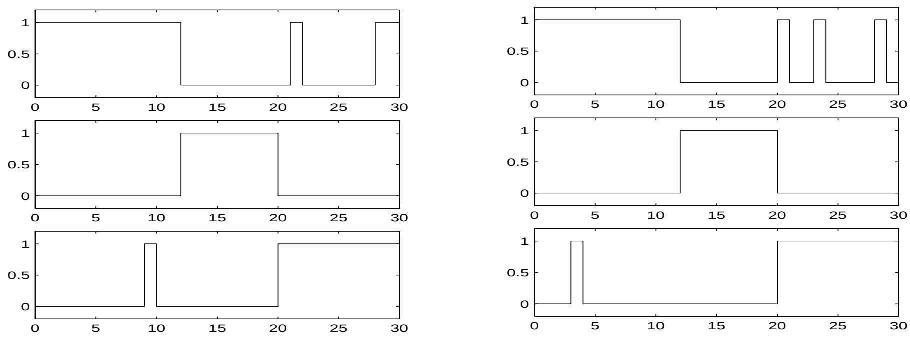

Fig. 6. Indicators of operating mode change.

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