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Evaluation of the Worst-Case Performance of Active Filters, using Robust Control Ideas

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Abstract. A methodology is proposed to determine the worst case effect that tolerances in components of active filters might have on its performance. The approach, based on the use of the Structured Singular Value, is shown to provide repeatable and non-probabilistic strict bounds on filter performance, allowing the designer to focus on worst case "limit of performance" comparisons when selecting filter structure, order and component ratings.

Keywords: Active Filter Analysis, Robust Control, Structured Singular Value.

1 Introduction

This paper considers a new type of performance analysis for active electronic filter circuits. The widespread mass manufacture of these circuits in all manner of different applications raises the question of how to ascertain what effect the unavoidable variation in component values will have on the transfer functions of active filters [11]. The standard practice in electronic engineering is to use simulation based solutions to this problem, using software packages like PSPICE [15]. Unfortunately this approach is not sufficient to calculate the worst case system response. For example, worst case upper and lower bounds on filter gain in response to any possible (bounded) variation in component values allows *guaranteed* achievement of certain design specifications.

A branch of robust control theory is used here to address this problem in a rigorous manner. This paper shows an algorithm to transform this problem to a Robust Control problem that can be solved using the structured singular value μ (studied, for example, in [2, 3, 16, 12]), to determine bounds on the maximum deviation from nominal behaviour that can occur in a filter transfer function at any specified frequency of interest. The principal advantage in using the proposed approach is that the problem is now solved in a repeatable and non-probabilistic fashion. The use of the Structured Singular Value requires that filter sensitivity problems are recast as equivalent system stability questions. Moreover, that combination of components which will result in the worst case filter performance can be computed reliably using readily available software.

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The proposed analysis technique also compares favourably with an approach based on the analysis of differential sensitivities, which is sometimes used in practice, because the proposed technique fully addresses the issue of interdependency or cross coupling effects between components in a repeatable and deterministic fashion. Moreover, while a differential sensitivity approach will yield a global worst case effect (i.e., for all frequencies), the proposed approach provides information on the worst case combination of uncertain elements on a frequency by frequency basis.

Section II introduces the main ideas and considers the issues involved in formulating the filter sensitivity problem so that the Structured Singular Value can be brought to bear on it, illustrating how filter transfer functions need to be correctly configured so that Robust Control ideas can be applied. In section III a selection of the results that have been obtained is presented. The results considered here provide new insights on an engineer's ability to guarantee certain worst case performance specifications for a particular design.

2 Problem Formulation

This section explains how the question of filter sensitivity is arranged so that μ-theory can be brought to bear on it. The methodology used is parallel to that presented by the authors in [7] to address a similar question for passive filters.

For simplicity, but without loss of generality, a Cascade approach for filter design will be used, as it is the most common method to design filters [11]: it is based on the series connection of first or second-order sections whose component values can then be easily calculated. Using a Cascade approach greatly simplifies the construction of higher order filters, so they are frequently used in practice. Accordingly, the filter structures studied in this paper are designed using cascade methods. The operational amplifiers are supposed to be bandwidth-limited. This bandwidth limitation, associated with the use of practical op-amp circuits, can be readily included within the proposed approach, as it will discussed later.

A filter transfer function is considered with n uncertain parameters $\Delta_1, \ldots, \Delta_n$ embedded within it. Although uncertain, these parameters are constrained in size to lie within a certain set of values, i.e., a capacitor is allowed to be no more than 20% outside its nominal value. Let $\mathcal{D}$ denote the set of all bounded perturbations to the nominal filter transfer function, which is problem specific: each $\Delta_1, \ldots, \Delta_n$ can be thought of as a set of real perturbations to the ideal parameter values, and will be viewed as an n-tuple $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_n)$. The system obtained when each $\Delta_i = 0$ corresponds to the **nominal** or **unperturbed** filter transfer function. By convention, and without loss of generality, the bounds on each parameter may be normalized. Thus, each Δ_i can assume any value in the interval $[-1, +1]$. This leaves a family of systems, one system for every permissible perturbation $\Delta \in \mathcal{D}$ applied to the nominal system. This family of systems fully represents the effect that uncertainty can have on a nominal filter transfer function.

2.1 The Diagonal Perturbation Formulation (DPF)

The first step is to “extract” the uncertainty that is embedded within $\mathcal{D}$. The extracted uncertainty can then be viewed as an external Δ acting on the nominal system, $\tilde{H}(s)$, as shown in Fig. 3. Then, the nominal system is what is left behind when this uncertain Δ has been extracted. The diagonal structure of Δ ensures that each Δ_i is associated with one uncertain element only. This rearranged representation of the original system is called its *Diagonal Perturbation Formulation (DPF)*. and is denoted as $G(s, \Delta)$. A tutorial example of the steps involved in generating the DPF for a passive, first order low pass RC filter is presented in [7]. It has already been shown that any linear transfer function can be expressed in terms of its DPF [5].

2.2 Algorithm to determine the Diagonal Perturbation Formulation

The procedure to determine the DPF for a linear filter is now discussed. A DPF representation for higher order filters is constructed by the Cascade connection of lower order sections, so only low order sections are considered.

Step 1: Generate the block diagram of Fig. 3 that represents the nominal input/output behaviour of the filter in a canonical form. This form can be achieved by inspection or through the use of signal flow graph techniques.

Step 2: Incorporate the effect of uncertainty into the block diagram of Fig. 3.

(i) Uncertainty in the numerator of an uncertain component X can be represented by a feedforward arrangement given by

$$X = X_0(1 + \alpha_X \Delta_X)$$

where X_0 is the nominal value $\alpha_X \in \mathbb{R}^+$ is a weighting that corresponds to a component tolerance and $\Delta_X \in \mathbb{R}$ is a real uncertain parameter, that varies between -1.0 and +1.0.

(ii) The effect of uncertainty associated with components that appear in the denominator of a transfer function can be represented by a feedback arrangement: In this way the impedance of a component such a as a capacitor with nominal value C_0 is given by

$$\frac{1}{sC_0(1 + \alpha_C \Delta_C)}$$

where $\alpha_C \in \mathbb{R}^+$ and $\Delta_C \in \mathbb{R}$, varying between -1.0 and +1.0.

(iii) For an active block (for example, the real operational amplifier, which is bandwidth-limited), a similar approach can be used: consider the nominal transfer function

$$Z_0(s) = \frac{K}{1 + \frac{s}{B}},$$

where K and B represent the nominal gain and bandwidth, respectively. Thus, uncertainty can be introduced using a feedforward scheme for K and a feedback scheme for B. As there can be gain and phase effects associated with a variation in bandwidth $\Delta_B \in \mathbb{C}$ is a complex parameter.

Step 3: Once uncertainty has been added to each component, these uncertainties can be "extracted" from the nominal system representation by associating an extra input/output pair with each Δ which are now assumed to be located on the main diagonal of an external system block. This completes the conversion of the system representation into the DPF type scheme of Fig. 3.

2.3 Formal Statement of the Filter Robustness Problem

A precise definition of worst case filter performance is now given using the terminology that has been introduced thus far. The maximum transfer function gain from input signal $a(s)$ to output signal $b(s)$ of the system represented by Fig. 3 may be written as

$$\max_{\Delta\in\mathcal{D}} \left|\frac{b(s)}{a(s)}\right| \;=\; \max_{\Delta\in\mathcal{D}} |G(s,\Delta)| \;=\; G_{max}(w,\Delta). \tag{1}$$

Similarly, the minimum transfer function gain is given by

$$\min_{\Delta\in\mathcal{D}} \left|\frac{b(s)}{a(s)}\right| \;=\; \min_{\Delta\in\mathcal{D}} |G(s,\Delta)| \;=\; G_{min}(w,\Delta), \tag{2}$$

which will determine the minimum possible filter response for all values of frequency. Solution of the optimisation problems in eqns. (1) and (2) will determine two distinct Δ's.

2.4 Application of the Robust Performance Theorem

The filter sensitivity problem is now recast as an equivalent robust stability question. The argument to be used is based on an application of the **Robust Performance Theorem** [3], which is now discussed briefly. Consider the system of Fig. 4: There are n uncertain parameters $\Delta_1,\ldots,\Delta_n$ which correspond to variations in component values. A "fictitious" uncertain parameter $k\Delta_f(k>0,\Delta_f\in\mathbb{C}$, will be used as a bound on the gain of the filter, as illustrated in Fig. 4. Here, k is a positive real scalar, and Δ_f is viewed as unknown, but constrained to have modulus ≤ 1 at each frequency, i.e.,

$$\Delta_f\in\mathcal{D}_f \;\text{ where }\; \mathcal{D}_f=\{\Delta_f\in\mathcal{C}\mid,\ |\Delta_f(s)|\le 1\}$$

In this fashion the gain and phase effects of perturbations to a nominal filter transfer function are fully addressed. The fictitious term $k\Delta_f$ can be included as an additional element in the diagonal matrix $\widetilde{\Delta}$. Therefore,

$$\widetilde{\Delta}=\begin{pmatrix} k\Delta_f & 0\\ 0 & \Delta\end{pmatrix}$$

and the set of uncertainties considered is then

$$\widetilde{\mathcal{D}}\equiv\{\widetilde{\Delta}|\widetilde{\Delta}=\begin{pmatrix} k\Delta_f & 0\\ 0 & \Delta\end{pmatrix},\Delta\in\mathcal{D},\Delta_f\in\mathcal{D}_f\}.$$

Now there is an equivalent representation of $\widetilde{H}(s)$ whose input/output pairs are connected exclusively by one $\widetilde{\Delta}$ block. Consider this self contained two block feedback loop. Let the filter gain k be fixed and given for the moment. By considering the Nyquist stability criterion, it is clear that if there is a $\Delta \in \mathcal{D}$ for which $|G(s,\Delta)| \geq k^{-1}$, then there is a $\Delta_f \in \mathcal{D}_f$ for which the system is unstable (having a loop gain ≥ 1). Conversely, if $|G(s,\Delta)| < k^{-1}$, for all $\Delta \in \mathcal{D}$, then the system is stable for all $\Delta_f \in \mathcal{D}_f$ (having a loop gain < 1 for every permissible perturbation). Thus, the maximum possible "size" of $|G(s,\Delta)|$ is bounded by k^{-1} if and only if a certain system is robustly stable. As k is increased, the first value of k for which this feedback system may become unstable corresponds to the largest possible $|G(s,\Delta)|$ being k^{-1}. There will therefore be a distinct value for k where instability occurs at each frequency. This is an important feature in that one frequency is decoupled from another. Information of this kind means that the engineer can now think in frequency response terms, an obvious benefit in the present context of filter design.

The Structured Singular Value μ Consider the arrangement of Fig. 2: at frequency ω The Structured Singular Value of Matrix Transfer Function $\tilde{H}(s)$, with uncertainty $\tilde{\Delta}(s)$ can be defined [2] as follows:

$$\mu\left(\tilde{H},\tilde{\Delta},\omega\right) \equiv \left(\min_{\tilde{\Delta}\in\tilde{\mathcal{D}}}\{k|\det(1+\tilde{\Delta}(j\omega)\tilde{H}(j\omega))=0\right)^{-1}. \tag{3}$$

Thus, the smallest k that will cause instability in the closed loop system at that frequency is μ^{-1}. This k should now be viewed as a stability margin for the simultaneous perturbation of a parameterized system by n uncertain components. μ-analysis exploits the *a priori* knowledge that exists about the internal structure of the uncertainty in a system and treats it in a worst case sense.

It should be noted that the proposed approach is based on the fact that the performance parameter k is seen only by the top row of the matrix $\tilde{H}(s)$ in the arrangement of Fig. 4. The other rows of $\tilde{H}(s)$ are acted on by real Δ's which are fixed in size. This is an example of the so called *Skewed Structured Singular Value* problem [14, 13]. Solution of this problem involves defining

$$K_m = \begin{pmatrix} k & 0 \\ 0 & I \end{pmatrix}$$

and locating the smallest k that will make det$(I + K_m\tilde{\Delta}(s)\tilde{H}(s))$ = 0 at any given frequency. Thus, the following definition can be given:

$$\mu_S(\tilde{H},\tilde{\Delta},\omega) \equiv (\min_{\tilde{\Delta}\in\tilde{\mathcal{D}}}\{k|\det(1+K_m\tilde{\Delta}(j\omega)\tilde{H}(j\omega))=0\})^{-1}. \tag{4}$$

2.5 Computing Worst Case Filter Sensitivity

At this point, the problem at hand is to compute bounds $G_{max}(w,\Delta)$ and $G_{min}(w,\Delta)$ on the magnitude of the filter frequency response, over a frequency range of interest. It is now shown how these functions reduce to an evaluation of skewed-μ for certain matrices, when appropriate constructions are used, which is the main result in this paper.

Theorem 1. *The frequency response of filter $G(\omega, \Delta)$ at each frequency ω is bounded by*

$$G_{min}(\omega, \Delta) \leq G(\omega, \Delta) \leq G_{max}(\omega, \Delta) \tag{5}$$

where

$$G_{min}(\omega, \Delta) = \lim_{\epsilon \to \infty} \mu_S \left(\frac{\epsilon}{1 + \epsilon \tilde{H}}, \tilde{\Delta}, \omega \right) \tag{6}$$

$$G_{max}(\omega, \Delta) = \mu_S \left(\tilde{H}, \tilde{\Delta}, \omega \right). \tag{7}$$

In these equations:

- $\tilde{H}$ *corresponds to the nominal system in the Diagonal Perturbation formulation of* $G(s, \Delta)$,
- Δ *corresponds to the uncertain matrix in the Diagonal Perturbation Formulation of the filter (in this case, it is a diagonal matrix composed of the uncertainty of the components of the filter:* $\Delta \in \mathbb{R}^{n \times n}$, *with* $|\Delta_{ii}| \leq 1$ *and* $\Delta_{ij} = 0$ *when* $i \neq j$*).*
- Δ_f *corresponds to the fictitious uncertainty:* $\Delta_f \in \mathbb{C}, |\Delta_f| \leq 1$.

Proof. **Maximum Gain**: Based on the definition of the Skewed Structured Singular Value, and the considerations in previous sections, the maximum filter gain at frequency ω can be computed from

$$\begin{aligned} G_{max}(\omega, \Delta) &= \left(\min_{\tilde{\Delta} \in \tilde{\mathcal{D}}} \{ k | \det(1 + K_m \tilde{\Delta}(j\omega) \tilde{H}(j\omega)) = 0 \} \right)^{-1} \\ &= \mu_S \left(\tilde{H}, \tilde{\Delta}, \omega \right) \end{aligned} \tag{8}$$

which corresponds to the smallest gain k for which the system may be unstable (i.e., not robustly stable). Thus, $G_{max}(s)$ yields the largest possible magnitude of $H(s, \Delta)$ at each value of frequency.

Minimum Filter Gain

A difficulty with the application of the robust performance theorem arises when attempting to determine the minimum possible filter gain. The theorem is only of use in finding the maximum allowable gain before a loop goes unstable. However, the minimum gain can be determined from $G(s, \Delta)$ by placing it on the feedback path of a suitable system, such as the one illustrated in Fig. 5. Again, each of the representations in the figure are equivalent.

Let $\epsilon \in \mathbb{R}$ be a constant gain term, which can be assumed to be arbitrarily large. Evaluating the closed loop response of the feedback system Fig. 5(i) yields

$$\frac{\bar{b}(s)}{\bar{a}(s)} = \frac{\epsilon}{1 + G(s, \Delta)\epsilon}$$

Thus, for large ϵ

$$\frac{\bar{b}(s)}{\bar{a}(s)} \approx \frac{1}{G(s, \Delta)}$$

Then if ϵ is large enough, the gain of the feedback system will be a maximum when the gain of $G(s, \Delta)$ is a minimum. Since the system has now been rearranged into a a robust stability question of standard form, the Robust Performance Theorem can be applied. Thus,

$$G_{min}(\omega, \Delta) \equiv (\min_{\tilde{\Delta}\in\tilde{\mathcal{D}}}\{k|\det(1+K_m\tilde{\Delta}(j\omega)\frac{\epsilon}{1+\epsilon\tilde{H}}) = 0\})^{-1} \quad (9)$$

$$= \mu_S\left(\frac{\epsilon}{1+\epsilon\tilde{H}}, \tilde{\Delta}, \omega\right)$$

where such a k corresponds to the minimum gain possible from $G(s, \Delta)$.

Remark 1. Bounds on worst case performance can now be obtained by calculating the skewed-μ using already available algorithms, implemented within off-the-self software. This calculation also gives the worst-case uncertainties, which can be transformed easily to worst-case component values.

Remark 2. Although it is known that considering uncertainties that are real lead to some computational difficulties, the fact the the fictitious gain is complex is known to improve the numerical behaviour, giving tight bounds on the skewed-μ value [4].

Remark 3. From a theoretical point of view it is not necessary to approximate $G_{min}(\omega, \Delta)$ using the fictitious feedback with ϵ: $G^{-1}(\omega, \Delta)$ can be approximated using a suitable Linear Fractional Transformation [17]. However the proposed numerical method is simpler to apply and gives a tight approximation, with good numerical behaviour.

3 State-space μ-analysis

It has been shown that Filter robustness analysis reduces to a question of checking the value for $\mu_{\mathcal{K}}(G(s, \Delta))$ over the closed right-half-plane (where $G(s)$ is a stable system and $\mathcal{X}_{\mathcal{K}}$ represents an appropriate uncertainty set depending on the problem at hand, be it maximum or minimum filter gain). This approach can be computationally intensive and an appropriate frequency vector for the analysis stage requires consideration. A sweep of frequency means that it is impossible to guarantee a true upper bound on filter gain. Moreover as skew$\backslash$real μ may not be a smooth function of the problem data [4], including the worst case frequency itself as part of the skew-μ problem is intuitively appealing. The question of focussing the analysis machinery on the particular frequency region where the worst case occurs is therefore now considered. The authors suggest that a so called "state-space μ" approach be applied to this problem to determine a guaranteed upper bound on worst case filter performance. The next subsection provides, without proof, the appropriate state space μ result for the filter performance question. For the necessary proofs and a more extensive digest of the literature on this subject the reader should consult [10].

3.1 State-space μ

The development of state-space μ is based on the fact that a transfer function can be expressed as an LFT of a constant matrix on the frequency variable. Given a transfer function $G(s)$, its equivalent state space representation is given by

$$G(s) = C(sI_p - A)^{-1}B + D = \frac{1}{s}I_p * \hat{G} \tag{10}$$

where $\hat{G}$ is the constant matrix

$$\hat{G} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{11}$$

and p is the order of the state-space. State space μ analysis recasts the filter analysis question as a μ computation on the static matrix $\hat{G}$ where the frequency variable $\frac{1}{s}I_p$ is included as an uncertain δ_i. The standard grid computation of μ is then transformed to a test on a static matrix with frequency explicitly represented as an uncertain (real) parameter within that static matrix. Unlike classical μ-analysis, this approach allows a frequency interval to be selected *a priori* where $\omega \in [\underline{\omega}, \overline{\omega}]$, by using the transformation

$$T = \begin{bmatrix} 0 & I_p \\ \frac{1}{2}I_p & \frac{1}{2}I_p \end{bmatrix}$$

and introducing the parameters

$$\omega_0 = \frac{1}{2}(\overline{\omega} + \underline{\omega})$$
$$\alpha_\omega = \frac{1}{2}(\overline{\omega} - \underline{\omega}).$$

The following result extends work first presented in [10] so that the evaluation of filter gain reduces to an application of the main loop theorem and a computation of $\mu(T * \hat{G})$.

Theorem 2. *Suppose that $G(s)$ has all of its poles in the open left-half-plane and let $\beta > 0$. Given a minimal state-space representation of $G(s)$ and the uncertainty structure $\mathcal{X}_{\hat{\mathcal{K}}}$, then for all $\Delta \in \mathcal{G}(X_{\mathcal{K}})$ with $\|\Delta\|_\infty \leq \beta$, the perturbed closed-loop system is uniformly stable if and only if*

$$\mu_{\hat{\mathcal{K}}}(T * \hat{G}) < 1$$

where

$$T * \hat{G} = \begin{bmatrix} j\alpha_\omega \mathcal{A}_3^{-1} & \sqrt{\frac{1}{\beta}}\mathcal{A}_3^{-1}B \\ -j\sqrt{\frac{1}{\beta}}\alpha_\omega C\mathcal{A}_3^{-1} & -\frac{1}{\beta}\left(C\mathcal{A}_3^{-1}B - D\right) \end{bmatrix} \tag{12}$$

with

$$\mathcal{A}_3^{-1} = (A - j\omega_0 I_p)^{-1}$$

A full proof of this result is presented in [10]. This result provides a one shot static matrix μ test for the general filter performance problem that yields an upper bound on filter gain over a user defined region of frequency, $\omega \in [\underline{\omega}, \overline{\omega}]$.

4 Examples

Some examples are now presented to give a summary of the results that can be obtained using the approach presented in this paper:

4.1 Filter Analysis

A common way of constructing active filters is to use the Sallen-Key methodology. The transfer function for a typical Sallen-Key section is

$$\frac{b(s)}{a(s)} = K\frac{Z_2Z_4}{Z_1Z_2 + Z_3 + Z_4(1-K) + Z_2Z_3 + Z_4}.$$

Note that $K = 1 + \frac{R_A}{R_B}$ is the standard gain of a non-inverting operational amplifier and $Z_1, \ldots, Z_4$ represents the impedance of the associated simple passive components. For example, we can consider the second order low pass design shown in Fig. 1, which has the transfer function

$$\frac{b(s)}{a(s)} = \frac{K}{R^2C_1C_2s^2 + R(2C_2 + (1-K)C_1)s + 1}.$$

This filter then has 5 uncertain components (C_1, C_2, R, R_A and R_B).

The standard practice in electrical engineering to evaluate the effect of uncertainty is the use of Monte-Carlo techniques. As it is well known, Monte-Carlo analysis methods yield a result that underestimates the worst-case. Fig. 6 illustrates how the proposed approach provides an upper bound on worst case filter performance for a real example: Fig. 6 shows how the gap between these bounds can become quite large for higher order filters. In this example a Monte-Carlo approach, which randomly chooses 100 sets of components, radically underestimates worst case filter performance on an 8^{th} order Band Pass Sallen-Key filter. It can be seen from this how the proposed approach gives the designer a bound on the "correct" side from a limit of performance perspective.

4.2 Filter Design

Another design that is frequently used in active filter synthesis is the Rausch structure, which uses multiple feedback loops to reduce the number of design parameters in the filter. A second order low pass Rausch structure is shown in Fig. 2 and has the following transfer function:

$$\frac{b(s)}{a(s)} = -\frac{1}{R^2C_1C_2s^2 + 3RC_2s + 1}$$

Although this filter has only three design parameters (C_1, C_2 and R), in practice the three resistances R do not have the same value, due to component tolerance, so in fact there are five uncertain parameters. The technique proposed in this paper can take this fact into account, by working with the wiring diagram through the Diagonal Perturbation Formulation.

As an example, the second order low pass sections of Fig. 1 and 2, with component tolerances of $\pm 20\%$ are considered. Fig. 7 shows how the proposed techniques confirms that a Sallen-Key filter is less sensitive to uncertainty than its Rausch and simple feedback counterparts in terms of improved worst case performance. This is due to the use of the gain controlled feedback stage in the filter where $K = 1 + \frac{R_1}{R_f}$. When $R_f >> R_1$, the sensitivity of the design to component variation is greatly reduced. In Fig. 8 the proposed technique demonstrates how component variation can limit the benefits that can be derived from an increase in filter order for fourth and eighth order Sallen-Key low pass filters that have been constructed using a series connection of second order sections. Fig. 9 illustrates how the proposed approach quantifies the benefits in moving to higher tolerance components, giving frequency specific information about the worst case effect of moving from 10% to 5% toleranced components on the second order Sallen-Key low pass design in Fig. 1.

4.3 State-Space μ

A summary of the results obtained using the new μ based analysis technique on a normalised 4th order Butterworth active low pass filter designed using an similar methodology to that outlined above is now presented. Two second order sections are placed in series and an LFT is generated using the `sysic` scripting routine from the μ-analysis toolbox.

Fig. 10 illustrates how the μ based analysis (dotted lines) provides upper bounds on filter performance. Here, the full line represents the nominal 4th order filter response. To highlight how a state space μ approach yields a safer upper bound by incorporating the frequency at which the worst case will occur explicitly within the analysis consider Fig. 11. Here the maximum deviation from the nominal for the 4th order filter response is calculated using both μ Toolbox and state space μ techniques. The μ-Toolbox ”upper” bound provides filter response information only at a user supplied vector of frequencies. The state space μ upper bound *guarantees* that no higher value of filter gain is possible within this range of frequency. The cost of this security is potential conservatism within the upper bound that is returned. The optimisation-based lower bound on state space μ gives a good estimate of the worst case frequency, (in this case, an irrational number). Moreover the worst case combination of filter components that is returned is not dominated by the filter performance I/O pair in the analysis. This feature is intuitively appealing. In this example the frequency where the worst case occurs is determined during the first iteration of the μ algorithm, to be 0.994 radians per second, correct to three decimal places.

5 Conclusions

This paper has shown how robust control theory, can be used to obtain limits of performance for active RC filters, where the components used in their manufacture are subject to variation. The technique is based on mapping the worst-case performance of a filter to an equivalent Robust Stability problem where component uncertainty is represented by structured (real and complex) perturbations to a nominal filter transfer function.

The proposed methodology gives a repeatable non-probabilistic upper bound on worst case filter performance. It fully addresses the question of cross-coupling effects on a frequency by frequency basis. The proposed procedure is applicable quite generally to any method for designing linear filters. As an example it has been shown how Sallen-Key filters provide the lowest sensitivity to variation in component values.

The results illustrate how this new approach provides a very useful tool for a comparative analysis of the worst case performance of different active filter configurations:

bounds on filter frequency response are given, so that limit of performance decisions can be made reliably when moving to higher order filters or using lower tolerance components.

A state space μ approach gives the designer a safe upper bound so that limit of performance decisions can be made reliably when moving to higher order filters or using lower tolerance components.

In conclusion, this paper has illustrated a very useful analysis tool from both an educational and a practical engineering design perspective. Although the technique has been presented for active filter in Cascade configuration, the technique can be used for other kind of filters.

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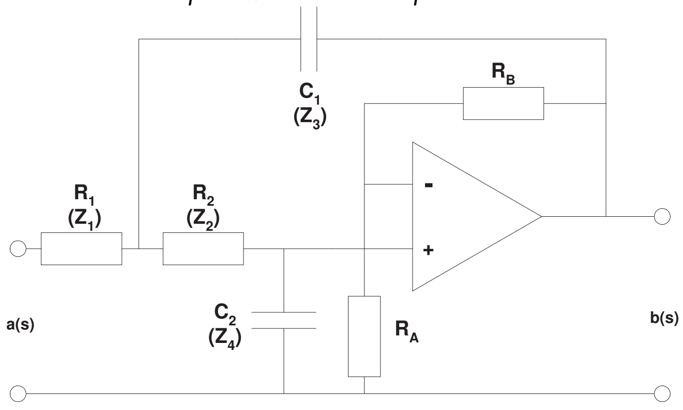

Fig. 1. Second Order low pass Sallen Key section

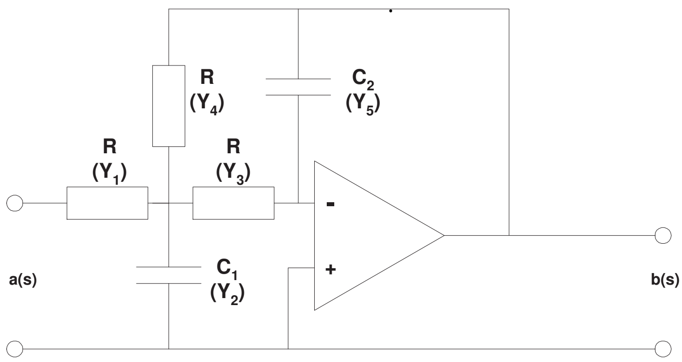

Fig. 2. Second Order low pass Rausch section

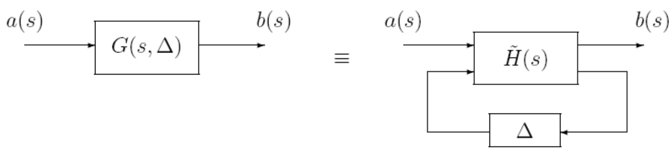

Fig. 3. The Diagonal Perturbation Formulation (DPF).

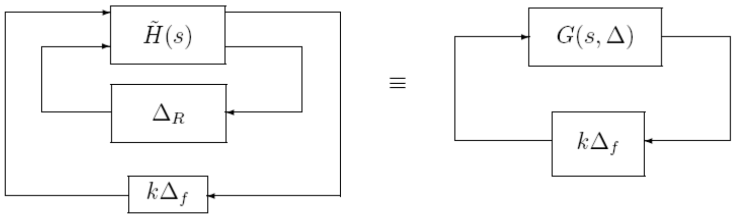

Fig. 4. Incorporating fictitious performance parameter into the DPF.

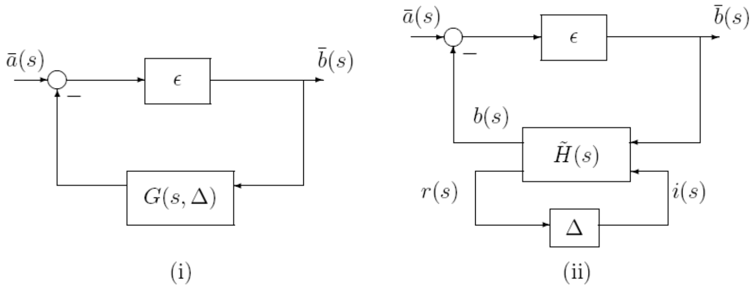

Fig. 5. Determination of the minimum gain from $G(s, \Delta)$.

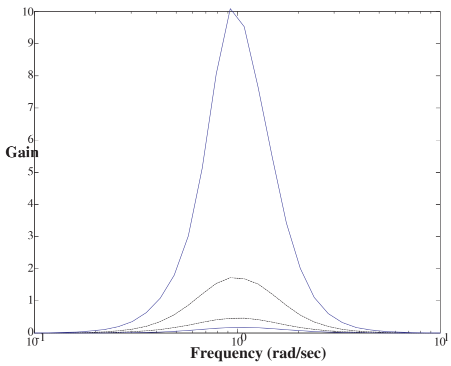

Fig. 6. Monte-Carlo (dashed) vs. Worst-case performance determined with the proposed technique.

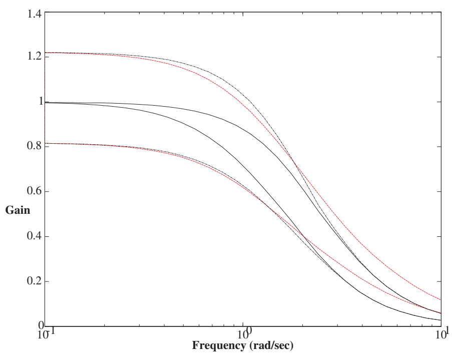

Fig. 7. Comparison, using the proposed approach, of Sallen-Key (full), Rausch (dashed) and Simple Feedback (dash-dot) filters.

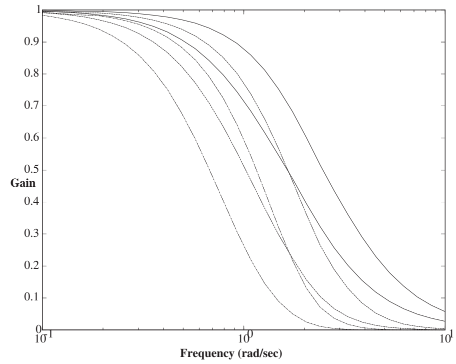

Fig. 8. Comparison, using the proposed approach, of worst-case performance of Sallen-Key filters of orders 4 (full), 6 (dashed) & 8 (dash-dot).

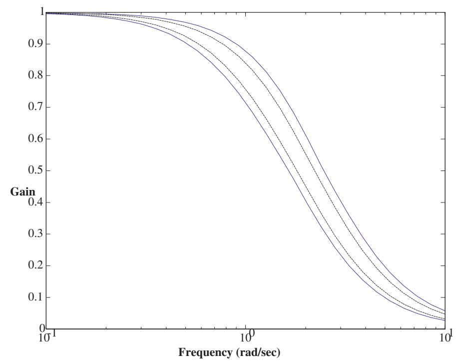

Fig. 9. Analysis of the benefit in moving to higher toleranced components (dashed: 5%, full: 10%).

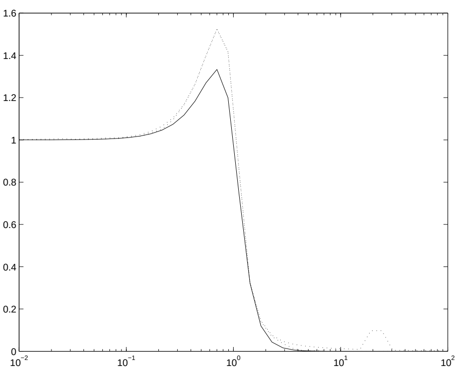

Fig. 10. 4th order Active Butterworth Filter Response: nominal vs. bounds on filter performance obtained with the proposed technique (dotted).

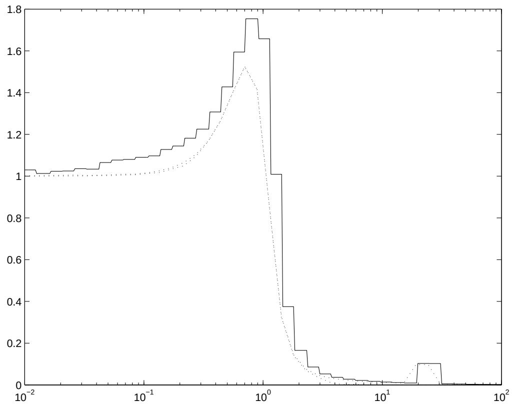

Fig. 11. State-Space μ vs. Robust Control Toolbox upper bounds on 4th order Butterworth filter performance.