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Numerical Approach to enlarging Domain of Attraction using LMI and Threshold Accepting Algorithms

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Abstract. *Asymptotic Stability region of nonlinear dynamical systems are estimated by homogeneous polynomial Lyapunov function, which include quadratic Lyapunov functions as a special case. We use an optimization strategy based on Linear Matrix Inequality (LMI), to compute the Domain of Attraction (DA). The main contribution consist in a preliminary estimation of the general form of this region obtained by the trajectory reversed Method. The parameters of Lyapunov function are computed by combining Threshold Accepting Algorithms and LMI. The proposed approach yields a larger stability region for polynomial systems than an existing method does. We demonstrate the performances of the proposed approach with two examples. The first is an academic example. The second example is a second order nonlinear chemical reactor CSTR. Simulations studies are developed to show the efficiency of the proposed method.*

Keywords. *Stability, Polynomial Lyapunov Function, Threshold Accepting Algorithms, Domain of Attraction, LMI.*

1. Introduction

When we control nonlinear dynamical systems, the asymptotic stability region of the corresponding equilibrium point is one of a criterion to evaluate control laws. The Domain of Attraction (DA) of equilibrium point that is the set of initial conditions from which the state converge to the equilibrium point itself [9]. Since finding the exact region of asymptotic stability is difficult or even impossible in general [9] and it does not admit attractable analytic representation in the most case. For this purpose we can use a Lyapunov Function [1, 2, 3]. Indeed, for a giving Lyapunov

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Functions providing local stability of the equilibrium point, the largest estimated of DA, whose shape is fixed by the Lyapunov Function itself, is defined as the largest level set of the Lyapunov included in the region where its derivative is negative [3, 4, 6].

Recently, there has been a lot of interest in the use of Polynomial Lyapunov Functions (PLFs) and Homogeneous Polynomial Lyapunov Functions (HPLFs) for extending and improving the results achieved through quadratic Lyapunov functions [2, 6]. In [3], the author proposes static nonlinear output feedback controllers enlarging the largest DA. The controllers proposed are polynomial in the measurable output.

The author propose to exploit relaxations based on sum of squares of polynomials in order to prove that the lower bound of the maximum achievable largest estimate of the region of attraction and a corresponding controller can be computed via a generalized eigenvalue problem. The advantage is that the problem is formulated as a quasi-convex LMI optimization. We propose here an alternative way to estimate the asymptotic stability region with good accuracy and tractable manner. To this end, we employ homogeneous polynomial Lyapunov functions [2, 6], which include quadratic Lyapunov functions as special case.

The purpose of this paper is proposed to enlarge the DA. The starting point of this work is the method developed in [3]. The objective is to improve the results by enlarging the DA that can be obtained by the last method. The objective is to improve the result by combining a Threshold Accepting Algorithm (TA) as an advanced optimization routine to LMI techniques in order to maximize the DA. Based on the Reverse Trajectory Method (RTM) one can estimate a preliminary maximal region of asymptotic stability and thereafter determine the parameter of the maximal Homogeneous Polynomial Lyapunov Function [6]. This will lead to an accurate definition of the initial values and the main constraints relating the required parameters of the investigated Lyapunov function. Note that the RTM is an important technique which offers the advantage of estimating bounded and unbounded asymptotic stability regions [8, 10]. Nevertheless, this method does not lead to an analytic expression of the estimated DA. It gives an interesting graphic representation of the shape of the DA.

In order to get an explicit expression of the estimated DA, we Lyapunov approach but with parameterized investigate a Homogeneous Polynomial Lyapunov function. Since the DA is related to the Lyapunov function, the idea consists in choosing the best matrix P in order to obtain the largest DA. This matrix is computed as solutions of an optimization problem. We propose to apply a Threshold Accepting Algorithm to solve this problem. We obtain by this way an optimal set matrix P , which is used to solve the LMI optimization problem derived from [3]. The main advantages of the TA are their robustness and efficiency in different environments covering various applications.

This article is organized as follows: In section 2 we describe the class of system considered and recall some basic notions. In section 3 we give a definition of reversed trajectory method and in section 4, we state our main results. In section 5, we illustrate our result with two examples. Finally, section 6 gives conclusions.

2. Problem Formulation and Preliminaries

2.1. Notation and representation Polynomial

The notation adopted in the paper is as follows: 0_n origin of R_n ; I_n identy matrix $n\times n$; A' : transpose of matrix A ; $A>0\,(A\geq 0)$: symmetric positive definite (semi definite) matrix A ; $A\otimes B$:Kronecker's product of matrices A and B ; $x^{\{\delta\}}\in R^{\varsigma(n,\delta)}$: vector containing all monomials of degree less or equal to δ in x but the constant term. For example $x^{\{\delta\}}=\left[x_1,x_2,\ldots,x_n,x_1^2,x_1x_2,\ldots,x_n^{\delta}\right]'$. Quantities $\varsigma(n,\delta)$ is given by

$$\varsigma(n,\delta)=\frac{(n+\delta)!}{n!\delta!}-1 \tag{1}$$

2.2. Problem Formulation

Consider the continuous-time polynomial system

$$\begin{cases}\dot{x}=f(x)+g(x)u\\ y=h(x)\end{cases} \tag{2}$$

Where f,g and h are polynomial functions such that $f(0)=0$. In the following, $x\in R^n$ is the state vector and $u\in R^p$ is the input vector. $y\in R^q$ is the measurable output. The equilibrium point of interest is the origin.
The control input is supposed to be a polynomial function of the form

$$u=U\phi(y) \tag{3}$$

where ϕ is a given polynomial function of the output and $U\in R^{p\times r}$ is a matrix belonging to the interval matrix

$$\mathbf{U}=\left\{U\setminus U_{i,j}\in\left(U_{i,j}^-,U_{i,j}^+\right),i=1,\ldots,p,\,j=,\ldots,r\right\} \tag{4}$$

Before proceeding further, we will give some preliminary results.

Let us consider $V(x)\in R$, a positive definite, radially unbounded and continuously differentiable function. The bounded set

$$\Omega(c) = \{x \in R^n / V(x) \le c\} \tag{5}$$

is an estimate of the DA if $\Omega(c) \subset D$ where $D = \{x \in R^n / \dot{V}(x,U) < 0\} \cup \{0\}$. The time derivative of $\dot{V}(x,U)$ along trajectory of system (2) is given by:

$$\begin{aligned} \dot{V}(x,U) &= \frac{\partial V(x)}{\partial x}\left(f(x) + g(x)U\phi(h(x))\right) \\ &= \frac{\partial V(x)}{\partial x} f(x) + \frac{\partial V(x)}{\partial x} g(x)U\phi(h(x)) \\ &= L_f V(x) + L_g V(x)U\phi(h(x)) \end{aligned} \tag{6}$$

$L_f V(x)$ (respectively $L_g V(x)$) is the Lie derivative of $V(x)$ along the polynomial vector $f(x)$ (respectively $g(x)$). In the following, we will denote by $L_{(g,U)} V(x) = L_g V(x)U\phi(h(x))$.

The largest estimate of the DA is given by $\Omega(c^*(U))$

where

$$c^*(U) = \inf_{x \in R^n} V(x) \qquad \text{such that} \quad \dot{V}(x,U) = 0 \tag{7}$$

for each matrix $U \in \boldsymbol{U}$. The optimal value of $c^*(U)$ is obtained by

$$c^* = \sup_{U \in u} c^*(U) \tag{8}$$

It has been proved in [3] that for any given $c \in R, c \le c^*$ if there exist $U \in \boldsymbol{U}$ and $s(x)$ a positive definite polynomial such that

$$\dot{V}(x,U) + (c - V(x))s(x) < 0 \tag{9}$$

The polynomial degrees of $V(x)$ and $\dot{V}(x,U)$ are $2\delta_V$ and δ_L respectively. We choose $s(x)$ degree to be $2\delta_s$ such that

$$\delta_s \ge \frac{\delta_L}{2} - \delta_V \tag{10}$$

It follows that the degree of the polynomial

$$t(x,U,c,s(x)) = \dot{V}(x,U) + (c - V(x))s(x) \tag{11}$$

is equal to $2\delta_m$ where $\delta_m = \delta_s + \delta_V$.

A Square Matricial Representation (SMR) and complete square matrix representation (CSMR) of polynomials (see for example [3]) are used in order to get an appropriate optimization problem. The CSMR provide all the possible representations of a polynomial in terms of a quadratic form.

The CSMR matrix of $t(x,U,c,s(x))$ is given by

$$T(\alpha,U,c,S)=D_f(\alpha)+D_g(U)+cW_1(S)-W_2(S) \qquad (12)$$

where $D_f(\alpha)$ is the CSMR of $L_f V(x)$, $D_g(U)$ is the SMR of $L_{(g,U)}V(x), W_1(S)$ and $W_2(S)$ are the SMR of $s(x)$ and $V(x)s(x)$.

The condition (9) with (12) implies that if

$$\hat{c}^* = \sup_{U\in u,\alpha,S>0} c \quad \text{Such that} \quad T(\alpha,U,c,S)<0 \qquad (13)$$

then $\hat{c}^* \le c^*$.

This leads to a non-convex problem (since c multiplies the parameters of S in $T(\alpha,U,c,S)$. The following Theorem is a reformulation of this problem as a generalized eigenvalue problem (GEVP), which enable to overcome this limitation.

Theorem 1: *([3]). The lower bound $\hat{c}^*$ is given by*

$$\hat{c}^* = \frac{-\lambda^*(U)}{1+\mu\lambda^*(U)} \qquad (14)$$

with $\lambda^*(U)$ solution of the following GEVP

$$\lambda^*(U) = \inf_{U\in u,\alpha,S>0,\lambda} \lambda$$
$$such\ that \begin{cases} 1+\mu\lambda>0 \\ U\in u \\ S>0 \\ \lambda W(S) > D_f(\alpha)+D_g(U)-W_2(S) \end{cases} \qquad (15)$$

where μ can be any positive scalar and

$$W(s)=K'\left(\begin{bmatrix}1 & 0\\ 0 & \mu V\end{bmatrix}\otimes S\right)K \qquad (16)$$

where $\otimes$ represents Kronecker's product and the matrix k satisfies.

$$\begin{bmatrix}1\\ x^{\{\delta_v\}}\end{bmatrix}\otimes x^{\{\delta_s\}} = Kx^{\{\delta_m\}} \qquad (17)$$

where

$$x^{\{\delta_m\}}\in R^{\varsigma(n,\delta_m)}, \alpha\in R^{\tau(n,\delta_m)}, x^{\{\delta_V\}}\in R^{\varsigma(n,\delta_V)}, x^{\{\delta_s\}}\in R^{\varsigma(n,\delta_s)} \qquad (18)$$

and

$$K\in R^{\varsigma(n,\delta_s)(\varsigma(n,\delta_V)+1)\times\varsigma(n,\delta_m)} \qquad (19)$$

it can be verified that the quantities $\varsigma(n,\delta_m)$ and $\tau(n,\delta_m)$ are given by

$$\varsigma(n,\delta_m)=\frac{(n+\delta_m)!}{n!\delta_m!}-1 \qquad (20)$$

$$\tau(n,\delta_m)=\frac{1}{2}\varsigma(n,\delta_m)\left(\varsigma(n,\delta_m)+1\right)-\varsigma(n,2\delta_m)+n \qquad (21)$$

3. The Reversing Trajectory Method

In this section, the reversing trajectory method [8] is recalled, having as a purpose to estimate the Region of Asymptotic Stability (RAS) through reversing the system trajectory flow. For a system of the form:

$$\dot{x}=F(x) \qquad (22)$$

the time reversing of this system (backward integration of (22)), turns out to consider the system :

$$\dot{x}=-F(x) \qquad (23)$$

The need, felt in engineering applications, to enlarge an initial arbitrary small estimation of the region of asymptotic stability [8], led to the general formulation of the reversing trajectory method.

The following Theorem provides sufficient conditions for the enlargement of the guaranteed stability region via the reversing trajectory method.

Theorem 2: *Given the autonomous system (23) with continuous right member, if the origin is asymptotically stable, there exists a positive definite Lyapunov function* $V(x)$ *such that:*

1) $B(0,R)=\{x\in R^n / V(x)<c\}$ *is simply connected with boundary* Γ_0,
2) $\dot{V}(x)<0,\ \forall x\in\{x\in R^n / V(x)<c\},\ x\neq 0$

then the region of asymptotic stability may be approximated arbitrarily well by means of a convergent sequence of simply connected domains generated by the backward integration technique, starting from the initial guaranteed stability region estimation $B(0,R)$.

The RAS estimation of the origin of system (23) using the reversing trajectory method involves the following five steps [8]:

1) Determine the equilibrium points of the system (different from the origin) and realize the local stability analysis of these points.

2) Determine an arbitrary small stability region $B(0,R)$ around each asymptotically stable equilibrium point in the space domain of interest.

3) Apply the backward integration of the system (23) at the asymptotically stable equilibrium point of step 2.

4) Perform forward and backward integration of the system (22) in the neighborhood of the other critical points in the domain of interest.

5) Derive an estimate of the region of asymptotic stability of the origin by means of topological considerations concerning the about the behavior of the obtained trajectories.

One of the objectives of the paper is to enlarge the RAS obtained by the method of [3]. The idea consists in taking the RAS obtained by this last method as the initial domain in the step 2 of the next main algorithm.

The second objective is to give an analytical expression of this RAS. This is not given by the reversing trajectory method, which finally give an interesting graphical representation of the RAS. In order to determine this expression of the RAS, we use Lyapunov function as it is presented in the next section.

4. Polynomial Lyapunov Function searching with Threshold Accepting Algorithms

The Threshold Accepting (TA) method is a variant of the classical simulated annealing algorithm, originally introduced by Dueck and Scheuer [5]. In this respect, TA abridged the simulated annealing procedure by excluding the element of probability in accepting inferior solutions. What is more, TA introduced a deterministic threshold and an inferior solution is accepted if its disparity with the existing solution is smaller than or equal to the threshold. The major components of TA are the functions that decide the lowering of the threshold during the course of the procedure, preventing criteria as well as the methods used to generate primary and neighbouring solutions. The main benefits of TA are its theoretic simplicity and its exceptional performance on various combinatorial optimization problems. On the assumption that X is the set of all feasible solutions of the problem, TA starts with a primary solution $X_0 \in X$ which may be generated randomly or used as a simple method. The method proceeds in an iterative manner. In each iteration, the algorithm determines if the new solution $X^{'}$ is less than the current solution X_c then the original one will be replaced by the new one; otherwise, another solution will be generated. If the current solution is less than the best solution so far X_b, then the best solution will be replaced by X_c.

4.1 The Main Result

The purpose of this section is to present a method to estimate analytically the RAS. We consider for this a polynomial Lyapunov function of the form:

$$V(x) = x^{\{\delta_V\}'} P x^{\{\delta_V\}} \tag{24}$$

where

$$P = P^T > 0 \tag{25}$$

We suppose that $\delta_V = 2$, we have:

$$P = \begin{bmatrix} p_1 & p_2 & 0 & 0 & 0 \\ p_2 & p_3 & 0 & 0 & 0 \\ 0 & 0 & p_4 & p_5 & 0 \\ 0 & 0 & p_5 & p_6 & p_7 \\ 0 & 0 & 0 & p_7 & p_8 \end{bmatrix} \tag{26}$$

Then

$$V(x) = p_1 x_1^2 + 2 p_2 x_1 x_2 + p_3 x_2^2 + p_4 x_1^4 + 2 p_5 x_1^3 x_2 + 2 p_6 x_1^2 x_2^2 + 2 p_7 x_1 x_2^3 + p_8 x_2^4 \tag{27}$$

4.2 The Main Result

This section is dedicated to the formulation of the final algorithm which leads to the DA estimation of the studied system:

Step 1: Apply the method given in [3] to estimate an initial DA.

Step 2: Apply the RTM and deduce the equation of the largest ellipsoid included in the obtained RAS.

Step 3: Generate randomly matrix P. Determine the positive definiteness of P. If P is not positive definite, go to **Step 3**.

Step 4: Compute the domain of attraction with the new matrix P.

Step 5: Accept solutions P and $\hat{c}^*$.

With this algorithm, we can maximize the region of asymptotic stability. Moreover the obtained solution is specified by the definition of a maximal quadratic Lyapunov function.

5. Illustrative Examples

Example 1

In order to illustrate the effectiveness of our results, we consider the following example, which has been considered in [3]:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 - x_1^2 - x_2^3 + x_1 u \\ \dot{x}_2 = -2x_2 - x_1^2 + u \\ y = x_1 \end{cases} \tag{28}$$

Our main goal is to show that the DA that we can estimate is larger than the one obtained in [3]. The proposed controller (3) is defined by

$$u = U\phi(y) \tag{29}$$

with:

$$\phi(y) = \begin{pmatrix} y \\ y^2 \end{pmatrix} \tag{30}$$

and

$$\mathcal{U} = \{U : u_i : -1 \le u_i \le 1,\ i = 1,2\} \tag{31}$$

When we apply the RTM described previously and defined in step 2, we obtain the figure 1 which represents the region of asymptotic stability of the studied system.

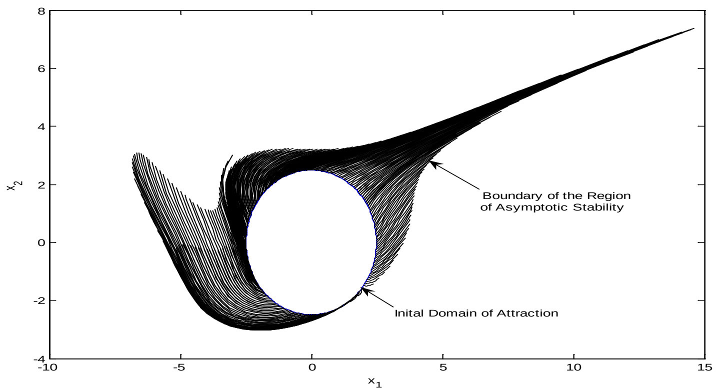

Fig.1. Region of Asymptotic Stability obtained by RTM.

According to the algorithm, proposed in paragraph 4, we employ a Homogeneous Polynomial Lyapunov function of the form:

$$V(x) = p_1 x_1^2 + 2p_2 x_1 x_2 + p_3 x_2^2 + p_4 x_1^4 + 2p_5 x_1^3 x_2 + 2p_6 x_1^2 x_2^2 + p_7 x_1 x_2^3 + p_8 x_2^4 \tag{32}$$

in order to find the shape of the RA. Since the degree δ_L of $\dot{V}(x,U)$ is 6, we can select $\delta_s = 1$ which implies that $m = 3$. Vectors $x^{\{\delta_s\}}$, $x^{\{\delta_V\}}$ and $x^{\{m\}}$ are selected as:

$$x^{\{\delta_s\}} = (x_1, x_2)^T, \quad x^{\{\delta_V\}} = (x_1, x_2, x_1^2, x_1 x_2, x_2^2)^T \tag{33}$$

and

$$x^{\{\delta_m\}} = \left(x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\right)^T \qquad (34)$$

which implies that:

$$D_f\left(p_{1,\ldots,8}\right) = \begin{bmatrix} -2p_1 & (p_1-3p_2) & -(p_1+p_2) & -(p_2+p_3) & 0 & 0 & 0 & 0 & 0 \\ (p_1-3p_2) & 2(p_2-2p_3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(p_1+p_2) & 0 & -4p_4 & 2p_4 & 0 & -2p_4 & -p_6 & -p_6 & -2p_8 \\ -(p_2+p_3) & 0 & 2p_4 & 6p_6 & (p_6-p_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (p_6-p_1) & 2(4p_8+p_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & -2p_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_6 & 0 & 0 & 0 & 0 & -2p_4 & 0 \\ 0 & 0 & -2p_8 & 0 & 0 & 0 & 0 & 0 & -p_6 \end{bmatrix}$$

$$D_g\left(U, p_{1,\ldots,8}\right) = \begin{bmatrix} 2p_2u_1 & p_3u_1 & p_1u_1+p_2u_2 & p_3u_2+p_2u_1 & 0 & 0 & 0 & 0 & 0 \\ p_3u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1u_1+p_2u_2 & 0 & 2p_1u_2 & p_6u_1+p_2u_2 & 0 & 2p_4u_1 & p_6u_2 & p_6u_1 & 2p_8u_2 \\ p_3u_2+p_2u_1 & 0 & p_6u_1+p_2u_2 & 0 & 2p_8u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_8u_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2p_4u_1 & 0 & 0 & 4p_4u_2 & 0 & 0 & 0 \\ 0 & 0 & p_6u_2 & 0 & 0 & 0 & 2p_6u_2 & 0 & 0 \\ 0 & 0 & p_6u_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2p_8u_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

and

$$W_2\left(S, p_{1,\ldots,8}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1p_1 & (s_2p_1+s_1p_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (s_2p_1+s_1p_2) & (s_1p_3+s_3p_1+4s_2p_2) & (s_2p_2+s_3p_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (s_2p_2+s_3p_2) & s_3p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1p_4 & s_2p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_2p_4 & (s_3p_4+s_1p_6) & s_2p_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2p_6 & (s_1p_8+s_3p_6) & s_2p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2p_8 & s_3p_8 \end{bmatrix}$$

$$W\left(S, p_{1,\dots,8}\right)=\begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu s_1 p_1 & \mu\left(s_2 p_1 + s_1 p_2\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu\left(s_2 p_1 + s_1 p_2\right) & \mu\left(s_1 p_3 + s_3 p_1 + 4 s_2 p_2\right) & \mu\left(s_2 p_2 + s_3 p_2\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\left(s_2 p_2 + s_3 p_2\right) & \mu s_3 p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu s_1 p_4 & \mu s_2 p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu s_2 p_4 & \mu\left(s_3 p_4 + s_1 p_6\right) & \mu s_2 p_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu s_2 p_6 & \mu\left(s_1 p_8 + s_3 p_6\right) & \mu s_2 p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu s_2 p_8 & \mu s_3 p_8 \end{bmatrix}$$

In step 3, we define the shape of the DA with the Homogeneous Polynomial Lyapunov function we perform this step of the algorithm, by considering the best estimation of parameters $p_1,\dots, p_8$.
The result of this step is

$$V\left(x\right)=60x_1^2 - 41x_1x_2 + 72x_2^2 + x_1^4 + x_1^2x_2^2 + 11x_2^4 = 898.2553$$

and is represented in figure 2.
This corresponds to the input $u = 0.77706x_1 - 0.4166x_1^2$

It is obvious that the final obtained domain of attraction is larger than the initial domain given by [3].

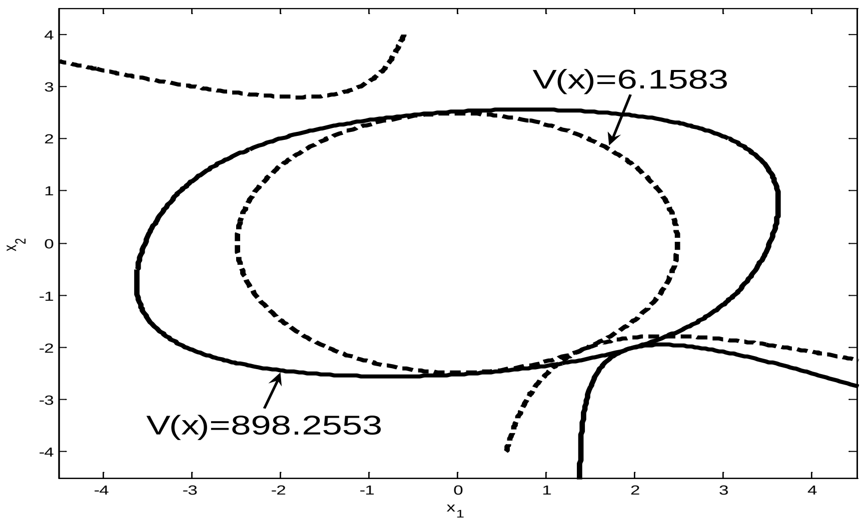

Fig.2. Solid line indicates the constraint $\dot{V}\left(x,U\right)=0$ and the boundary of the DA $V\left(x\right)=898.2553$ that we obtained via the method (with Threshold Accepting Algorithms); the dashed line indicates the constraint $\dot{V}\left(x,U\right)=0$ and $V\left(x\right)=6.1583$ obtained in [3].

Example 2

Let us consider the following consecutive reactions

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

that take place in an isothermal continuous stirred-tank reactor (CSTR) with constant volume. It is assumed that A→B has first order kinetics whereas B→C is a second-order chemical reaction. The dynamical nominal model of the CSTR can be written in the following form [7]:

$$\begin{cases} \dot{x}_1 = -2x_1 + u \\ \dot{x}_2 = x_1 - x_2 - 2x_2^2 \\ y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{cases} \tag{35}$$

The controller is supposed linear in y, that is $\phi(y) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

The proposed controller (3) is defined by

$$u = U\phi(y) \tag{36}$$

Set $\mathcal{U}$ is chosen as

$$\mathcal{U} = \{U : u_i : -1 \le u_i \le 1,\ i = 1,2\} \tag{37}$$

When we apply the RTM described previously and defined in step 2, we obtain the figure 3 which represents the region of asymptotic stability of the studied system.

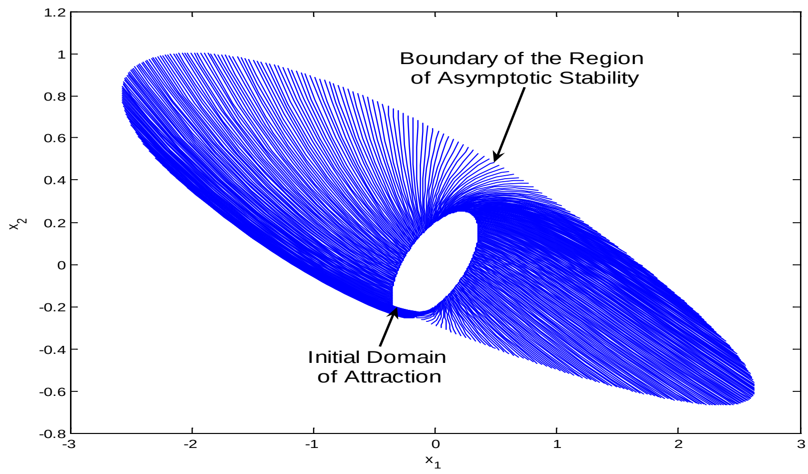

Fig.3. Region of Asymptotic Stability obtained by RTM.

Let us investigate the structure of the GEVP in (15) when the LF defining the LEDA shape is selected as

$$V(x) = p_1 x_1^2 + 2p_2 x_1 x_2 + p_3 x_2^2 + p_4 x_1^4 + 2p_5 x_1^3 x_2 + 2p_6 x_1^2 x_2^2 + p_7 x_1 x_2^3 + p_8 x_2^4 \quad (38)$$

in order to find the shape of the RA. Since the degree δ_L of $\dot{V}(x,U)$ is 6, we can select $\delta_s = 1$ which implies that $m = 3$. Vectors $x^{\{\delta_s\}}$, $x^{\{\delta_V\}}$ and $x^{\{m\}}$ are selected as:

$$x^{\{\delta_s\}} = (x_1, x_2)^T, \quad x^{\{\delta_V\}} = (x_1, x_2, x_1^2, x_1 x_2, x_2^2)^T \quad (39)$$

and

$$x^{\{\delta_m\}} = (x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)^T \quad (40)$$

which implies that:

$$D_f(p_{1,\dots,8}) = \begin{bmatrix} -2(2p_1 - p_2) & -(3p_2 - p_3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(3p_2 - p_3) & -2p_3 & 0 & -2p_2 & -2p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8p_4 & 2p_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2p_2 & 2p_6 & -12p_6 & -2p_8 & 0 & 0 & 0 & 0 \\ 0 & -2p_3 & 0 & -2p_8 & -4p_8 & 0 & -4p_6 & 0 & -4p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4p_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4p_8 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_g(U, p_{1,\dots,8}) = \begin{bmatrix} 2p_1 u_1 & (p_2 u_1 + p_1 u_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (p_2 u_1 + p_1 u_2) & 2p_2 u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4p_4 u_1 & 2p_4 u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2p_4 u_2 & 4p_6 u_1 & 2p_6 u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_6 u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

and

$$W_2(S,p_{1,\dots,8}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1p_1 & (s_2p_1+s_1p_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (s_2p_1+s_1p_2) & (s_1p_3+s_3p_1+4s_2p_2) & (s_2p_2+s_3p_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (s_2p_2+s_3p_2) & s_3p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1p_4 & s_2p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_2p_4 & (s_3p_4+s_1p_6) & s_2p_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2p_6 & (s_1p_8+s_3p_6) & s_2p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2p_8 & s_3p_8 \end{bmatrix}$$

$$W(S,p_{1,\dots,8}) = \begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu s_1p_1 & \mu(s_2p_1+s_1p_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu(s_2p_1+s_1p_2) & \mu(s_1p_3+s_3p_1+4s_2p_2) & \mu(s_2p_2+s_3p_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu(s_2p_2+s_3p_2) & \mu s_3p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu s_1p_4 & \mu s_2p_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu s_2p_4 & \mu(s_3p_4+s_1p_6) & \mu s_2p_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu s_2p_6 & \mu(s_1p_8+s_3p_6) & \mu s_2p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu s_2p_8 & \mu s_3p_8 \end{bmatrix}$$

In step 3, we define the shape of the DA with the Homogeneous Polynomial Lyapunov function we perform this step of the algorithm, by considering the best estimation of parameters $p_1,\dots,p_8$.
The result of this step is

$$V(x) = 70x_1^2 - 2x_1x_2 + 100x_2^2 + x_1^4 + 2x_1^2x_2^2 + 10x_2^4 = 24.2556$$

This corresponds to the input

$$u = -0.8069x_1 - 0.9320x_2$$

and is represented in figure 4.

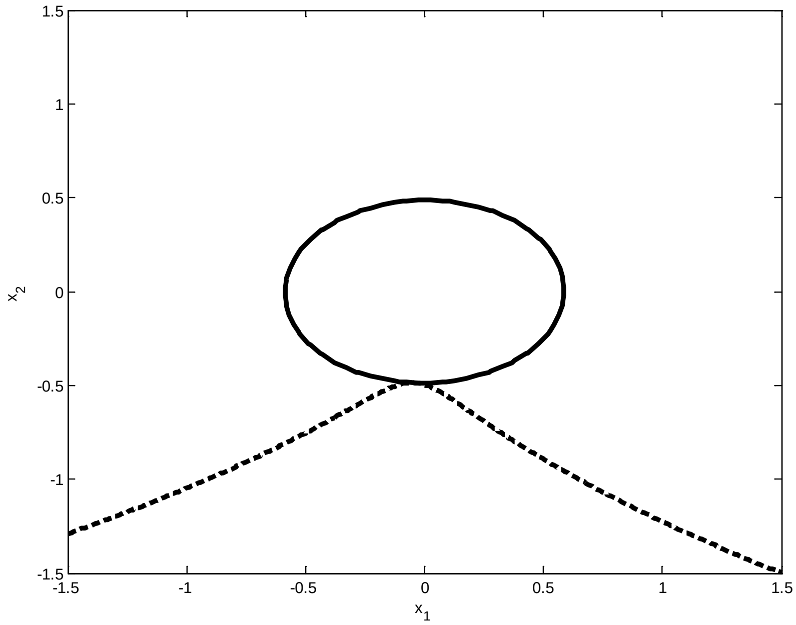

Fig.4. Solid line indicates boundary of the DA $V(x) = 24.2556$ that we obtained via the method (with Threshold Accepting Algorithms). The dashed line indicates the constraint $\dot{V}(x,U) = 0$.

6. Conclusion

In this paper, we have considered the problem of the enlargement of Domain of Attraction (DA) of nonlinear systems. The systems under consideration are polynomial and this is not restrictive since most of nonlinear dynamics can be approximated (using Taylor series) by polynomials. We use an optimization strategy based on Linear Matrix Inequality (LMI), to compute the Domain of Attraction. The main contribution consists in the determination of an explicit DA by using a parameterized Lyapunov function. The parameters are computed by combining Threshold Accepting Algorithms and an LMI. We also employ the trajectory reversing method to represent the shape of the largest of DA. Examples have illustrated the efficiency of the established results.

References

1. Hamidi, F. Jerbi, H. Aggoune, W. Djemai, M. and Abdelkrim, M. N.: Enlarging region of attraction via LMI-based approach and Genetic Algorithm. 1st International Conference on Communications, Computing and Control Applications, Hammamet Tunisia 2011.

2. Hamidi, F. Jerbi, H. and abdelkrim, M. N.: Searching candidate Lyapunov Function with Threshold Accepting Algorithm. 3rd International Conference on Computational Intelligence, Communication Systems and Networks, (CICSyN2011), Bali, Indonesia, 26-28 July, 2011.

3. Chesi, G.: Computing output feedback controllers to enlarge the domain of attraction in polynomial systems. IEEE Transaction on Automatic Control, vol. 49, pp. 1846-1850, 2004.

4. Chesi, G. Garulli, A. Tesi A. and Vicino, A.: Solving quadratic distance problems: an LMI approach. IEEE Transaction on Automatic Control. vol. 13, pp. 200-212, 2003.

5. Duech, D and Scheuer, T.: Threshold Accepting: A general purpose optimization algorithm appearing superior to simulated annealing. Journal of Computational Physics 90, pp. 161-175, 1990.

6. Fujisaki, Y, and Sakuwa. R.: Estimation of asymptotic stability regions via homogeneous polynomial Lyapunov functions. International Journal Control, vol. 79, no. 6, pp. 617-623, 2006.

7. Garcia-Sandoval, J.P. Gonzalez-Alvarez, V. Castillo-Toledo, B. and Pelayo-Ortiz, C.: Robust discrete control of nonlinear processes: Application to chemical reactor. Computers and Chemical Engineering, vol 32, pp.3246–3253, 2008.

8. Genesio, R. Tartaglia, M. and Vicino, A.: On the estimation of asymptotic stability regions: state of the art and new proposals. IEEE Transaction on Automatic Control, vol. AC 30, no. 8, pp. 747-755, 1985.

9. Khalil, H.K.: Nonlinear system. In Third Edition Upper Saddle River, NJ: Prentice Hall, 2001.

10. Loccufier, M. and Noldus, E.: A new trajectory Reversing Method for Estimating Stability Regions of autonomous Nonlinear Systems. In Nonlinear dynamics 21, pp. 265-288, 2000.