International Journal of Sciences and Techniques of Automatic control & computer engineering
IJ-STA, Volume 7, N° 1, April 2013, pp. 1844–1863.

Adaptive Sliding Mode Controller Design for a Class of Nonlinear Systems

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Abstract. *An adaptive sliding mode controller is presented for a class of Single Input Single Output (SISO) nonlinear systems with external disturbance. The designed controller, resulting from the combination of the Classical Sliding Mode Control technique (CSMC), the Proportional Integral controller (PI) and the Proportional Derivative controller (PD), can reduce the chattering phenomenon and guarantee the asymptotic convergence to zero of tracking errors and the boundedness of all signals in the closed-loop system. The parameters of the proposed controller, the adaptive laws and the robust control term are derived based on the Lyapunov stability analysis. Finally, an inverted pendulum system and two coupled tanks system are simulated to demonstrate the validity and the effectiveness of the suggested approach.*

Keywords: *Adaptive Control, Sliding Mode Control, nonlinear Single Input Single Output (SISO) systems.*

1. Introduction

In addition to external disturbances, many dynamical systems have nonlinearities and parametric disturbances. Therefore, the use of robust control is desirable.
In recent years, several researchers have been interested to Sliding Mode Control (SMC), known as a robust control strategy for nonlinear systems **[1]-[3]**.

This paper was recommended for publication in revised form by the editor Staff.
Edition: CPU of Tunis, Tunisia, ISSN: 1737-7749

Known for its simple structure, easy implementation and its robustness to external disturbances, the SMC has been a topic of great interest in control theory and represented a great potential for practical applications. However, it suffers from a main disadvantage: the chattering phenomenon, which is the high frequency oscillation of the controller output.

To overcome this problem, several methods of chattering reduction have been reported in literature such as in **[4]-[8]**. In **[9]**, the boundary layers approach can reduce this phenomenon. This method consists in replacing the discontinuous switching action by a continuous saturation function. This approach is generally appropriate for low disturbances and it requires an approximation of the term of discontinuity. Furthermore, in **[10]** an asymptotic observer can eliminate the chattering phenomenon. The application of such observer assumes that the unmodelled dynamics are completely unknown. To attain the same objective, another common method based on the high order SMC can be elaborated **[11]**. However, this method requires a complex calculation. Another way to solve this chattering problem is based on combining the SMC and intelligent controllers to approximate the switching control term such as in **[12]** and **[13]**. The free parameters of the adaptive fuzzy controller can be tuned online based on the Lyapunov approach **[14]**, **[15]**. In **[16], [17]** and **[18]**, the authors proposed a method to eliminate the chattering phenomenon by using an adaptive Proportional Integral controller (PI controller) for a SISO nonlinear system. The multi-input multi-output (MIMO) nonlinear systems are investigated in **[19]** and **[20]**.

In this paper, we tempt to reduce the chattering phenomenon and to ensure good tracking performances despite external disturbances. For this purpose, a Classical Sliding Mode is combined with an Adaptive Proportional Integral Controller as well as an Adaptive Proportional Derivative Controller. On the basis of the Lyapunov theory, we demonstrate that the proposed controller guarantees the convergence to zero of tracking errors and the boundedness of all signals in the closed-loop system.

The remaining of the paper is organized as follows. The problem formulation is described in Section 2. The classical sliding mode control is presented in Section 3. By combining the sliding mode with an adaptive proportional integral controller and an adaptive proportional derivative controller, an improved sliding mode controller and an adaptive sliding mode controller are designed to reduce the chattering phenomenon in Section 4 and 5. The simulation results of the inverted pendulum and the coupled tanks system are given, in Section 6, to show the effectiveness of the proposed control strategies. Finally, Section 7 gives a conclusion on the main works developed in this paper.

2. Problem formulation

Consider a class of nonlinear, Single Input Single Output (SISO), disturbed system:

$$\begin{cases} x^{(n)} = f(x) + g(x)u + d(t) \\ y = x_1 \end{cases} \tag{1}$$

where x is the state vector, $x = \left[x_1, x_2, \ldots, x_n\right]^T = \left[x, \dot{x}, \ldots, x^{(n-1)}\right]^T \in \Re^n$.
$u \in \Re$, $y \in \Re$ and $d(t)$ are respectively the input, the output and the unknown external disturbance of the system.
$f(x)$ and $g(x)$ are nonlinear functions.
We consider the following assumptions:

Assumption 1 *$g(x)$ is assumed to be controllable $g(x) \neq 0$, $\forall x$.*

Assumption 2 *The external disturbance $d(t)$ is bounded as $|d(t)| \leq D$.*

Our objective is to develop a control low allowing the output of the system "*y*" to follow a given signal "y_d" despite the external disturbances $d(t)$.

3. Classical sliding mode control

In order to develop the classical sliding mode approach for the SISO system two steps are required: the choice of sliding surface and the calculation of the control law.

3.1. Sliding surface

The sliding surface is defined by the following expression:

$$s = \sum_{j=1}^{r} \alpha_{r-j} e^{(r-j)} \text{ where } \alpha_{(r-1)} = 1, j=1..p \tag{2}$$

The parameters $\alpha_{(r-1)}, \ldots, \alpha_0$ are chosen such that all roots of $h(p) = p^{(r-1)} + \alpha_{(r-2)} p^{(r-2)} + \ldots + \alpha_1 \dot{p} + \alpha_0$ are in the left half plane. (Here p denotes the complex Laplace transform variable).
r represents the relative degree of the system. In fact, the sliding variable has a relative degree equal to one compared to the control law. This implies that the control law appears explicitly in the derivative of the sliding surface.
The tracking error is defined by:

$$e = y - y_d \tag{3}$$

3.2. Control law

The control law includes two terms: a continuous term known as the equivalent control u_{eq} and a switching term known as the discontinuous control u_{sw}.
The equivalent control is calculated for the derivative of the sliding surface is null and without external disturbances.

The time derivative of s can be obtained as:

$$\dot{s} = (x^{(n)} - y_d^{(n)}) + \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} \tag{4}$$

Substituting **(1)** in **(4)**, we obtain:

$$\dot{s} = f(x) + g(x)u - y_d^{(n)} + \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} \tag{5}$$

$\dot{s} = 0$, we are getting:

$$u_{eq} = g^{-1}(x)\left[-f(x) + y_d^{(n)} + \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} \right] \tag{6}$$

The switching law is given by the following expression:

$$u_{sw} = g^{-1}(x)\left[-\eta sign(s) \right], \eta > 0 \tag{7}$$

sign is the signum function.
Thus, the conventional sliding mode control law is given by:

$$u = g^{-1}(x)\left[-f(x) + y_d^{(n)} - \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} - \eta sign(s) \right] \tag{8}$$

Theorem 1. *Consider the class of SISO nonlinear systems* ***(1)****, if the control law* ***(8)*** *is applied, then the proposed control scheme guarantees the following properties:*

(i) The signals of the closed-loop system are bounded;
(ii) The tracking errors converge to zero.

Proof:
The Lyapunov function is chosen such as:

$$V(x) = \frac{1}{2} s^2(x) \tag{9}$$

Then, the time derivative of V is given by:

$$\dot{V} = s\dot{s} \tag{10}$$

According to **(4)**, **(8)** and **(10)**:

$$\dot{V} = s\left[f(x) + g(x)g(x)^{-1}\left(-f(x) + y^{(n)}{}_d - \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} - \eta sign(s) \right) - y_d^{(n)} + \sum_{i=2}^{n} \alpha_{(i-2)} e^{(i-1)} \right] \tag{11}$$

Consequently, the time derivative of V is written:

$$\dot{V} = -\eta s sign(s) = -\eta |s| < 0 \tag{12}$$

Using Barbalat's lemma **[18]**, we can see that the sliding surface converges asymptotically to zero in finite time despite the external disturbances.

4. Improved sliding mode control

Based on the classical sliding mode, we noticed that the presence of the signum function in the term u_{sw} leads to the chattering phenomenon which can excite the high frequency dynamics. In order to reduce this phenomenon and to achieve the control objective, we decided to substitute the term of discontinuity by an adaptive Proportional Integral (PI) controller. The gains of the proposed controller are adjusted online according to the desired performances.

The expression of PI, included in the control law, is written as follows:

$$u_{PI} = k_p s + k_i \int_{t_0}^{t} s(\tau) d\tau \tag{13}$$

where k_p and k_i are the control gains adjusted online from an adaptive law.

Remark. *From **(13)**, we notice that the proportional integral term depends on the sliding surface and consequently of all the dynamics of the tracking error.*

The adaptive PI term derived from **(13)** can be rewritten as:

$$u_{PI} = \rho\left(s\middle|\theta_\rho\right) = \theta^T{}_\rho \Theta(s) \tag{14}$$

where θ_ρ is the adjustable parameters vector given by $\theta_\rho = \left[k_p \ k_i\right]^T$ and $\Theta^T(s) = \left[s(t) \ \int_0^t s(t)\right]$ is the regressive vector.

Let us define the following variables:

Case1: $s > 0$

$$\theta^*_{\rho_1} = \arg\min_{\theta_{\rho_1} \in \Omega_{\rho_1}} (\sup_{s \in \mathbb{R}} \left|\rho\left(s\middle|\theta_{\rho_1}\right) - \eta\right|), \eta > 0 \tag{15}$$

Where Ω_{ρ_1} denotes the set of suitable bound on θ_{ρ_1} and $\eta\,\mathrm{sgn}(s)$ is the discontinuous term of the conventional sliding mode control.

Remark. *In this approach, the idea is to approximate a discontinuous term by a continuous term, which leads to an approximation error given by the following expression:*

$$\omega_{PI_1} = -\rho\left(s\middle|\theta^*_{\rho_1}\right) + \eta \tag{16}$$

Case2: $s < 0$

$$\theta^*_{\rho_2} = \arg\min_{\theta_{\rho_2} \in \Omega_{\rho_2}} (\sup_{s \in \mathbb{R}} \left|\rho\left(s\middle|\theta_{\rho_2}\right) + \eta\right|), \eta > 0 \tag{17}$$

Where Ω_{ρ_2} denotes the set of suitable bound on θ_{ρ_2}.

$$\omega_{PI_2} = -\rho\left(s\left|\theta^*_{\rho_2}\right.\right) - \eta \quad \textbf{(18)}$$

We define:

$$W_{PI} = \omega_{PI_i}, i = 1, 2 \quad \textbf{(19)}$$

The parameter approximation error:

$$\tilde{\theta}_{\rho} = \theta^*_{\rho_i} - \theta_{\rho}, i = 1, 2 \quad \textbf{(20)}$$

To achieve the control objective, we need to establish a control law that forces the trajectories of system status to reach and remain on the sliding surface despite the presence of external disturbances. We suggest adding a term of robustness u_1 in order to cancel the effect of the error of approximation.

$$u_1 = \hat{W}_{PI} \quad \textbf{(21)}$$

$\hat{W}_{PI}$ is the estimated of W_{PI} to be determined yet.

The control law is written as follows:

$$u = g^{-1}(x)\left[-f(x) + \ddot{y}_d - \alpha_1\dot{e} - \rho\left(s\left|\theta_{\rho}\right.\right) + u_1\right] \quad \textbf{(22)}$$

The parameter vector θ_{ρ} is adjusted online by the following adaptive laws:

$$\dot{\theta}_{\rho} = -\gamma_{\rho} s\Theta(s) \quad \textbf{(23)}$$

$$\dot{\hat{W}}_{PI} = -\gamma_{PI} s \quad \textbf{(24)}$$

where $\gamma_{\rho} > 0$ and $\gamma_{PI} > 0$ are the adaptation gains.

The main result of the improved sliding mode control proposed is summarized in the following theorem:

Theorem 2. *Consider the class of SISO nonlinear systems* ***(1)****, if the control law* ***(22)*** *is applied, where the terms u_{PI} and u_1 are respectively given by* ***(14)*** *and* ***(21)****. The parameters θ_{ρ} and $\hat{\omega}_{PI}$ are respectively adjusted on-line by applying the adaptation laws* ***(23)*** *and* ***(24)****, then the proposed control scheme guarantees the following properties:*

(i) The signals of the closed-loop system are bounded;
(ii) The tracking errors converge to zero.

Proof:

Let us consider the following Lyapunov function:

$$V = \frac{1}{2}s^2 + \frac{1}{2\gamma_{\rho}}\tilde{\theta}^T_{\rho}\tilde{\theta}_{\rho} + \frac{1}{\gamma_{PI}}(\tilde{W}^T_{PI}\tilde{W}_{PI}) \quad \textbf{(25)}$$

We define

$$\tilde{W}_{PI} = W_{PI} - \hat{W}_{PI} \quad \textbf{(26)}$$

The time derivative of s is given by:

$$\dot{s} = f(x) + g(x)u - \ddot{y}d + \alpha_1 \dot{e} + d$$

$$\dot{s} = f(x) + g(x)g^{-1}(x)\left[-f(x) + \ddot{y}d - \alpha_1 \dot{e} - \rho\left(s\middle|\theta_\rho\right) + u_1 + d\right]$$

$$\dot{s} = -\rho\left(s\middle|\theta_\rho\right) + u_1 + d \tag{27}$$

In this work, W_{PI} is assumed to be unknown. That is why they will be estimated on-line by using suitable adaptive laws deduced from the stability analysis in the Lyapunov sense.

Knowing that:

$$\begin{cases} \dot{\tilde{\theta}}_\rho = -\dot{\theta}_\rho \\ \dot{\tilde{W}}_{PI} = -\dot{\hat{W}}_{PI}, \; W_{PI} = 0 \; [21] \end{cases} \tag{28}$$

The time derivative of V is given by:

$$\dot{V} = s\dot{s} + \frac{1}{\gamma_\rho}\tilde{\theta}_\rho^T \dot{\tilde{\theta}}_\rho + \frac{1}{\gamma_{PI}}(\tilde{W}_{PI}^T \dot{\tilde{W}}_{PI}) \tag{29}$$

By substituting **(27), (28)** in **(29)**, we obtain:

$$\dot{V} = s(-\rho\left(s\middle|\theta_\rho\right) + u_1 + d) - \frac{1}{\gamma_\rho}\tilde{\theta}_\rho^T \dot{\theta}_\rho - \frac{1}{\gamma_{PI}}(\tilde{W}_{PI}^T \dot{\hat{W}}_{PI})$$

$$\dot{V} = su_1 - s\rho\left(s\middle|\theta_\rho\right) + s\rho\left(s\middle|\theta^*_\rho\right) - s\rho\left(s\middle|\theta^*_\rho\right) - \frac{1}{\gamma_\rho}\tilde{\theta}_\rho^T \dot{\theta}_\rho - \frac{1}{\gamma_{PI}}(\tilde{W}_{PI}^T \dot{\hat{W}}_{PI}) + sd$$

$$\dot{V} = \tilde{\theta}_\rho\left(s^T\Theta(s) - \frac{1}{\gamma_\rho}\dot{\theta}_\rho\right) + s\hat{\omega}_{PI} - s\rho\left(s\middle|\theta^*_\rho\right) - \frac{1}{\gamma_{PI}}(\tilde{W}_{PI}^T \dot{\hat{W}}_{PI}) + sd$$

$$\dot{V} = \tilde{\theta}_\rho\left(s^T\Theta(s) - \frac{1}{\gamma_\rho}\dot{\theta}_\rho\right) + s\left[\hat{\omega}_{PI} - \left(\omega_{PI} - \eta sign(s)\right)\right] - \frac{1}{\gamma_{PI}}(\tilde{W}_{PI}^T \dot{\hat{W}}_{PI}) + sd \tag{30}$$

By substituting **(23)**, **(24)** in **(30)**, we get:

$$\dot{V} \le -\eta s sign(s) + sd \tag{31}$$

$$\dot{V} \le -\eta\left|s\right| < 0, \quad \forall \eta > D \tag{32}$$

Using Barbalat's lemma **[18]**, we can see that the sliding surface converges asymptotically to zero in finite time despite the external disturbances.

5. Adaptive sliding mode control

To ensure the control robustness and to reduce the rapprochement phase to the sliding surface, we replaced, in the following section, the adaptive proportional integral controller by an adaptive proportional derivative controller with an integral surface. In

fact, the derivative action, by compensating the inertia due to dead time, accelerates the response of the system and improves the stability of the closed loop by allowing fast oscillations due to the appearance of a disturbance or a sudden change of the reference signal. Thus, it ensures a faster convergence to the sliding surfaces.
The integral sliding surface is defined by the following expression:

$$s = e^{(r-1)} + \sum_{j=2}^{r-1} \alpha_{r-j} e^{(r-j)} + k_1 \int_0^t e(\tau) d\tau \qquad \textbf{(33)}$$

The parameters $\alpha_{(r-2)}, \dots, k_1$ are chosen such that all roots of $h(p) = p^{(r-1)} + \alpha_{(r-2)} p^{(r-2)} + \dots + \alpha_1 \dot{p} + k_1$ are in the left half plane.
The behavior of the added proportional derivative term is similar to a proportional integral derivative without increasing the number of parameters to be adjusted compared to the previous adaptive proportional integral term.
Then, the expression of the proportional derivative term is written as follows:

$$u_{PD} = k_p s(t) + k_d \frac{d}{dt} s(t) \qquad \textbf{(34)}$$

where k_p and k_d are the control gains adjusted online from an adaptive law.
The adaptive PD term derived from **(34)** can be rewritten as:

$$u_{PD} = \rho\left(s \middle| \theta_\rho\right) = \theta^T{}_\rho \Theta(s) \qquad \textbf{(35)}$$

where θ_ρ is the adjustable parameters vector given by $\theta_\rho = \left[k_p \;\; k_d\right]^T$ and $\Theta^T(s) = \left[s(t) \;\; \frac{ds(t)}{dt}\right]$ is the regressive vector.
Let us define the following variables:
Case1: $s > 0$

$$\theta^*_{\rho_1} = \arg \min_{\theta_{\rho_1} \in \Omega_{\rho_1}} (\sup_{s \in \mathbb{R}} \left|\rho\left(s \middle| \theta_{\rho_1}\right) - \eta\right|), \eta > 0 \qquad \textbf{(36)}$$

Where Ω_{ρ_1} denotes the set of suitable bound on θ_{ρ_1} and $\eta\, \mathrm{sgn}(s)$ is the discontinuous term of the conventional sliding mode control.
The approximation error is given by the following expression:

$$\omega_{PD_1} = -\rho\left(s \middle| \theta^*_{\rho_1}\right) + \eta \qquad \textbf{(37)}$$

Case2: $s < 0$

$$\theta^*_{\rho_2} = \arg \min_{\theta_{\rho_2} \in \Omega_{\rho_2}} (\sup_{s \in \mathbb{R}} \left|\rho\left(s \middle| \theta_{\rho_2}\right) + \eta\right|), \eta > 0 \qquad \textbf{(38)}$$

Where Ω_{ρ_2} denotes the set of suitable bound on θ_{ρ_2}.

$$\omega_{PD_2} = -\rho\left(s \middle| \theta^*_{\rho_2}\right) - \eta \qquad \textbf{(39)}$$

We define:

$$W_{PD} = \omega_{PD_i}, i = 1,2 \quad \textbf{(40)}$$

The parameter approximation error:

$$\tilde{\theta}_{\rho} = \theta^*_{\rho_i} - \theta_{\rho}, i = 1,2 \quad \textbf{(41)}$$

Same as the previous improved control law, the robust control law can be written as follows:

$$u = g^{-1}(x)\left[-f(x) + \ddot{y}_d - \alpha_1\dot{e} - k_1 e - \rho\left(s\left|\theta_{\rho}\right.\right) + u_1\right] \quad \textbf{(42)}$$

where

$$\begin{cases} \rho\left(s\left|\theta_{\rho}\right.\right) = \theta^T_{\rho}\Theta(s) \\ u_1 = \hat{W}_{PD} \end{cases} \quad \textbf{(43)}$$

where $\hat{W}_{PD}$ is the estimated of W_{PD} to be determined yet.

The parameter vector θ_{ρ} is adjusted online by the following adaptive laws:

$$\dot{\theta}_{\rho} = -\gamma_{\rho} s\Theta(s) \quad \textbf{(44)}$$

$$\dot{\hat{W}}_{PD} = -\gamma_{PD} s \quad \textbf{(45)}$$

where $\gamma_{\rho} > 0$ and $\gamma_{PD} > 0$ are the adaptation gains.

The main result of the robust adaptive sliding mode control proposed is summarized in the following theorem:

Theorem 3. *Consider the class of SISO nonlinear systems* ***(1)****, if the control law* ***(42)*** *is applied, where the terms* u_{PD} *and* u_1 *are respectively given by* ***(35)*** *and* ***(43)****. The parameters* θ_{ρ} *and* $\hat{\omega}_{PD}$ *are respectively adjusted on-line by applying the adaptation laws* ***(44)*** *and* ***(45),*** *then the proposed control scheme guarantees the following properties:* *(i) The signals of the closed-loop system are bounded;*
(ii) The tracking errors converge to zero.

Proof:
Let us consider the following Lyapunov function:

$$V = \frac{1}{2}s^2 + \frac{1}{2\gamma_{\rho}}\tilde{\theta}^T_{\rho}\tilde{\theta}_{\rho} + \frac{1}{\gamma_{PD}}(\tilde{W}^T_{PD}\tilde{W}_{PD}) \quad \textbf{(46)}$$

Let us define:

$$\tilde{W}_{PD} = W_{PD} - \hat{W}_{PD} \quad \textbf{(47)}$$

In this work, W_{PD} is assumed to be unknown. That is why they will be estimated online by using suitable adaptive laws deduced from the stability analysis in the Lyapunov sense.
The time derivative of s is given by:

$$\dot{s} = f(x) + g(x)u - \ddot{y}d + \alpha_1\dot{e} + k_1 e + d$$

$$\dot{s} = f(x) + g(x)g^{-1}(x)\left[-f(x) + \ddot{y}d - \alpha_1\dot{e} - +k_1 e - \rho\left(s\left|\theta_{\rho}\right.\right) + u_1 + d\right]$$

$$\dot{s} = -\rho\left(s\left|\theta_{\rho}\right.\right) + u_{1} + d \quad \textbf{(48)}$$

Knowing that:

$$\begin{cases} \dot{\tilde{\theta}}_{\rho} = -\dot{\theta}_{\rho} \\ \dot{\tilde{W}}_{PD} = -\dot{\hat{W}}_{PD},\ \dot{W}_{PD} = 0\ [21] \end{cases} \quad \textbf{(49)}$$

The time derivative of V is given by:

$$\dot{V} = s\dot{s} + \frac{1}{\gamma_{\rho}}\tilde{\theta}_{\rho}^{T}\dot{\tilde{\theta}}_{\rho} + \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\tilde{W}}_{PD}) \quad \textbf{(50)}$$

By substituting **(48)** and **(49)** in **(50)**, we obtain:

$$\dot{V} = s(-\rho\left(s\left|\theta_{\rho}\right.\right) + u_{1} + d) - \frac{1}{\gamma_{\rho}}\tilde{\theta}_{\rho}^{T}\dot{\theta}_{\rho} - \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\hat{W}}_{PD})$$

$$\dot{V} = su_{1} - s\rho\left(s\left|\theta_{\rho}\right.\right) + s\rho\left(s\left|\theta_{\rho}^{*}\right.\right) - s\rho\left(s\left|\theta_{\rho}^{*}\right.\right) - \frac{1}{\gamma_{\rho}}\tilde{\theta}_{\rho}^{T}\dot{\theta}_{\rho} - \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\hat{W}}_{PD}) + sd$$

$$\dot{V} = su_{1} + s\tilde{\theta}_{\rho}^{T}\Theta(s) - s\rho\left(s\left|\theta_{\rho}^{*}\right.\right) - \frac{1}{\gamma_{\rho}}\tilde{\theta}_{\rho}^{T}\dot{\theta}_{\rho} - \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\hat{W}}_{PD}) + sd$$

$$\dot{V} = \tilde{\theta}_{\rho}\left(s^{T}\Theta(s) - \frac{1}{\gamma_{\rho}}\dot{\theta}_{\rho}\right) + s\hat{\omega}_{PD} - s\rho\left(s\left|\theta_{\rho}^{*}\right.\right) - \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\hat{W}}_{PD}) + sd$$

$$\dot{V} = \tilde{\theta}_{\rho}\left(s^{T}\Theta(s) - \frac{1}{\gamma_{\rho}}\dot{\theta}_{\rho}\right) + s\left[\hat{\omega}_{PD} - \left(\omega_{PD} - \eta sign(s)\right)\right] - \frac{1}{\gamma_{PD}}(\tilde{W}^{T}_{PD}\dot{\hat{W}}_{PD}) + sd \quad \textbf{(51)}$$

By substituting **(44)** and **(45)** in **(51)**, we get:

$$\begin{aligned} \dot{V} &\leq -\eta s sign(s) + sd \\ \dot{V} &\leq -\eta s sign(s) + \left|s\right| D \\ \dot{V} &\leq -\eta\left|s\right| < 0,\quad \forall \eta > D \end{aligned} \quad \textbf{(52)}$$

Using Barbalat's lemma **[18]**, we can see that the sliding surface converges asymptotically to zero in finite time despite the external disturbances.

6. Simulation results

To illustrate the effectiveness of the proposed approaches, two systems are simulated: an inverted pendulum system and a two coupled tanks system.

6.1 Inverted Pendulum

Consider the inverted pendulum system shown in **Fig. 1**.

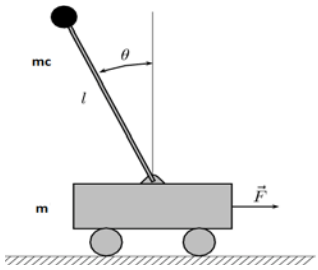

Fig. 1. Schematic of the inverted pendulum.

The structure of this pendulum is described by the following dynamic equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{g\sin x_1 - \dfrac{mlx_2^2\cos x_1 \sin x_1}{m_c m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)} + \dfrac{\dfrac{\cos x_1}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)} u + d \\ y = x_1 \end{cases} \tag{53}$$

where $x = [x_1, x_2]^T \in \Re^2$ the state vector, x is the angular of the pendulum with respect to the vertical line;
u is the applied force to move the cart (the control signal);
$u \in \Re$, $y \in \Re$ and $d(t)$ are respectively the input, the output and the unknown external disturbance of the system;
g: the acceleration of gravity;
l: the angular velocity of the pole with respect to the vertical axis;
m_c: the cart mass;
m: the pole mass.
The dynamic model of the system can be written in a compact form as:

$$\begin{cases} \ddot{x}_1 = f(x) + g(x)u + d \\ y = x_1 \end{cases} \tag{54}$$

where $f(x) = \dfrac{g\sin x_1 - \dfrac{mlx_2^2\cos x_1 \sin x_1}{m_c m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)}$ and $g(x) = \dfrac{\dfrac{\cos x_1}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)}$

The control objective is to force the system output "y" to track the desired trajectory "y_d". The desired trajectory is given by:

$$y_d = \frac{\pi}{10}(\sin(t) + 0.3\sin(3t)) \tag{55}$$

$$y_{dd} = \dot{y}_d = \frac{\pi}{10}\left(\cos(t) + 0.9\cos(3t)\right) \quad \textbf{(56)}$$

The parameter values of the inverted pendulum are presented in the following table:

Table 1. Values of the parameters of the inverted pendulum

m_c	1.5kg
M	0.2kg
L	1m
G	981 cm/s^2

The simulation results are shown in **Figs2-7.** The tracking curves of states (x_1 and x_2) and its reference signals (y_d and y_{dd}) are respectively displayed in **Figs 2-3**. Those curves prove the good performance of the proposed controller: The Adaptive Sliding Mode Controller (ASMC), in particularly the good tracking of the reference signals.
The evolution of the control signal, given in **Fig.5**, reveals the reduction of the chattering phenomenon comparing to the evolution of the control signal of the Classical Sliding Mode Control approach (CSMC) given in **Fig.6.** The evolution of the sliding surface (**Fig.7**) proves that the attractiveness of this surface is guaranteed.

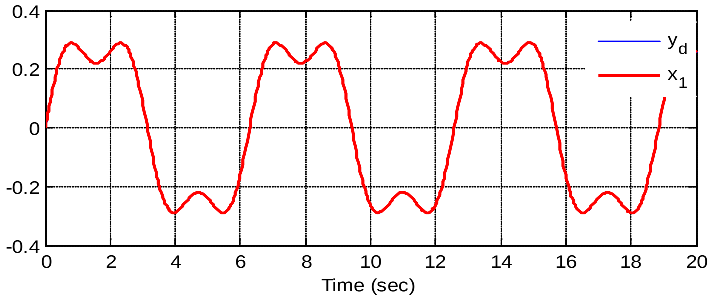

Fig. 2. The trajectories of the state vector x_1 and the reference signal y_d

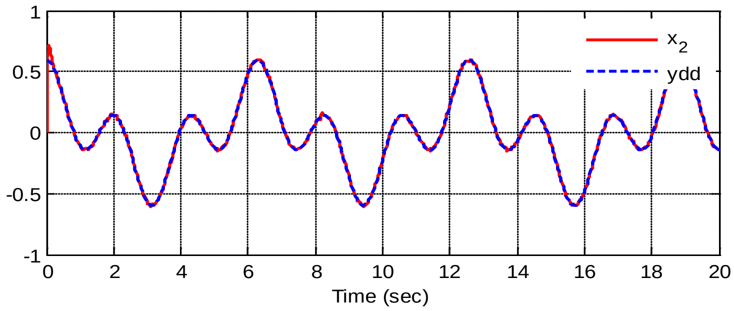

Fig. 3. The trajectories of the state vector x_2 and the reference signal y_{dd}

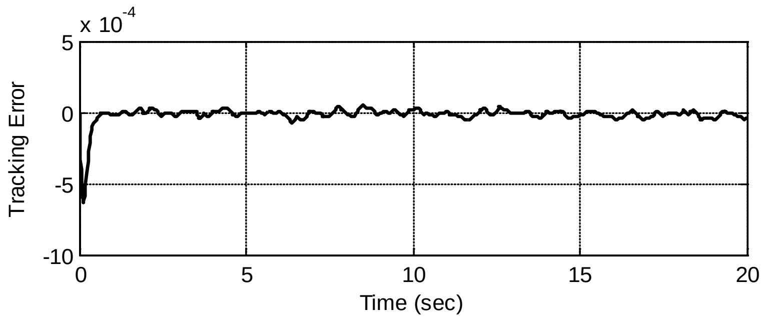

Fig. 4. Evolution of the tracking error.

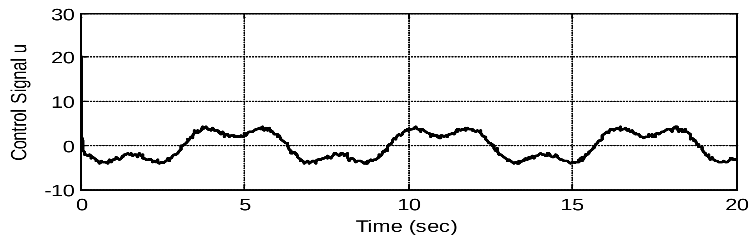

Fig. 5. Evolution of the control signal u (ASMC).

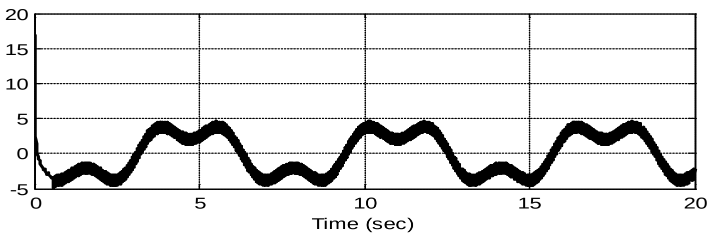

Fig. 6. Evolution of the control signal u (CSMC).

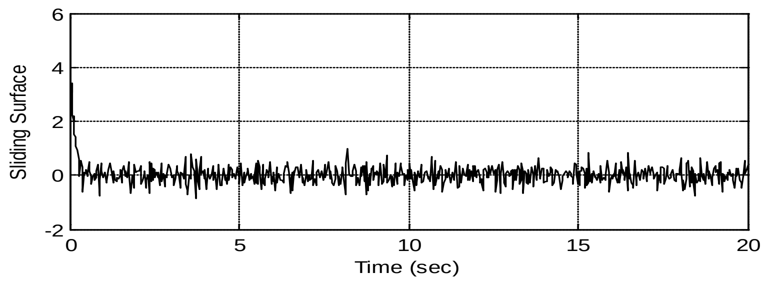

Fig. 7. Evolution of the classic sliding surface s.

6.2 Two coupled tanks system

In order to illustrate the previous concepts, let us consider a liquid level control of two tanks system shown in **Fig. 8**.

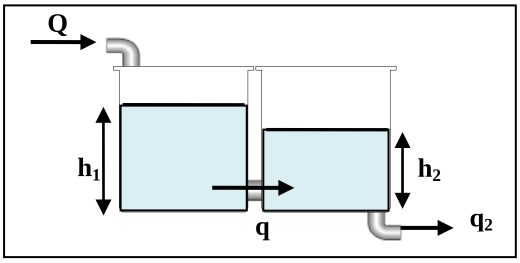

Fig. 8. Schematic of the coupled tanks system.

The hydraulic system with two tanks consists of two identical hold-up tanks coupled by an orifice. This orifice allows the liquid to flow into the second tank and hence out to a reservoir. The objective of the control is to adjust the inlet flow rate Q(t) so as to maintain the level in the second tank, $h_2(t)$ close to a desired set point level.

Let us take:

- h_1 (t) is the level in the first tank;
- h_2 (t) is the level in the second tank;
- q_1 (t) is the flow rate from tank 1 to tank 2;
- q_2 (t) is the flow rate out of tank 2;
- g is the gravitational constant;
- C_{12} is the area of the coupling orifice;
- C_2 is the area of the outlet orifice.

Applying Torricelli's law, the general model of the coupled tanks system can be written as follows:

$$\begin{cases} \dfrac{dh_1}{dt} = \dfrac{1}{C}(-q_1 + Q) \\ \dfrac{dh_1}{dt} = \dfrac{1}{C}(q_1 - q_2) \end{cases} \quad \textbf{(57)}$$

where

$$\begin{cases} q_1 = c_{12}\sqrt{2g\left|h_1 - h_2\right|}\,\mathrm{sgn}(h_1 - h_2) \\ q_2 = c_2\sqrt{2gh_2} \end{cases} \quad \textbf{(58)}$$

From **(54), (55)** can be written as:

$$\begin{cases} \dot{h}_1 = \dfrac{-c_{12}}{C}\sqrt{2g\left|h_1 - h_2\right|}\,\mathrm{sgn}(h_1 - h_2) + \dfrac{1}{C}Q \\ \dot{h}_2 = \dfrac{-c_{12}}{C}\sqrt{2g\left|h_1 - h_2\right|}\,\mathrm{sgn}(h_1 - h_2) - \dfrac{c_2}{C}\sqrt{2gh_2} \end{cases} \quad \textbf{(59)}$$

At equilibrium, for constant liquid level set point, the derivatives must be zero:

$$\begin{cases} \dfrac{-c_{12}}{C}\sqrt{2g\left|h_1-h_2\right|}\,\mathrm{sgn}(h_1-h_2)+\dfrac{1}{C}Q_0=0 \\ \dfrac{-c_{12}}{C}\sqrt{2g\left|h_1-h_2\right|}\,\mathrm{sgn}(h_1-h_2)-\dfrac{c_2}{C}\sqrt{2gh_2}=0 \end{cases} \quad \textbf{(60)}$$

where Q is the equilibrium inflow rate. The equilibrium inflow rate is always positive, for this reason $sign(h_1-h_2)\geq 0$ and $h_1\geq h_2$.

$$\begin{cases} \dot{h}_1=\dfrac{-c_{12}}{C}\sqrt{2g\left(h_1-h_2\right)}+\dfrac{1}{C}Q \\ \dot{h}_2=\dfrac{-c_{12}}{C}\sqrt{2g\left(h_1-h_2\right)}-\dfrac{c_2}{C}\sqrt{2gh_2} \end{cases} \quad \textbf{(61)}$$

In order to apply the control law **(39)**, we should rewrite the model **(61)** as **(1).** Therefore we consider the following transformations:

$z_1=h_2>0$, $z_2=h_1-h_2>0$, $Z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, $u=Q$, $a_1=\dfrac{c_2\sqrt{2g}}{C}$ and $a_2=\dfrac{c_{12}\sqrt{2g}}{C}$

The output of the coupled tanks system is taken to be the level of the second tank. Therefore, the dynamic model in **(61)** can be written as:

$$\begin{cases} \dot{z}_1=-a_1\sqrt{z_1}+a_2\sqrt{z_2} \\ \dot{z}_2=a_1\sqrt{z_1}-2a_2\sqrt{z_2}+\dfrac{1}{C}u \\ y=z_1 \end{cases} \quad \textbf{(62)}$$

The objective of the control scheme is to adjust the output $y(t)=z_1(t)=h_2(t)$ to a desired value H. It is easy to show using **(62)** that if $y(t)=z_1(t)$ is regulated to a desired value H, then $z_2(t)=h_1(t)-h_2(t)$ will be regulated to the value $\dfrac{a_1^2}{a_2^2}H$.

The dynamic model of the coupled tanks system is highly nonlinear. Therefore, we will define a transformation so that the dynamic model given in **(62)** can be transformed into a form that facilitates the control design.
We consider the following model:

$$\begin{cases} x_1=z_1 \\ x_2=-a_1\sqrt{z_1}+a_2\sqrt{z_2} \end{cases} \quad \textbf{(63)}$$

$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the transformation of z, $x=T(z)$. The inverse transformation $z=T^{-1}(x)$ is such:

$$\begin{cases} z_1 = x_1 \\ z_2 = \left(\dfrac{a_1\sqrt{x_1} + x_2}{a_2} \right)^2 \end{cases} \quad \textbf{(64)}$$

From **(62)** and **(64)**, the dynamic model of the system can be written in a compact form as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u \\ y = x_1 \end{cases} \quad \textbf{(65)}$$

where

$$f(x) = \frac{a_1 a_2}{2}\left(\frac{a_2\sqrt{x_1}}{a_1\sqrt{x_1} + x_2} - \frac{a_1\sqrt{x_1} + x_2}{a_2\sqrt{x_1}} \right) + \frac{a_1^{\;2}}{2} - a_2^{\;2}\,;\; g(x) = \frac{a_2^{\;2}\left(a_1\sqrt{x_1} + x_2\right)}{2C}$$

Taking into consideration the external disturbances, the system **(65)** can be rewritten:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u + d \\ y = x_1 \end{cases} \quad \textbf{(66)}$$

Finally, the dynamic model of the system can be written in a compact form as:

$$\begin{cases} \ddot{x}_1 = f(x) + g(x)u + d \\ y = x_1 \end{cases} \quad \textbf{(67)}$$

The objective of the control is to adjust the output $y(t) = z_1(t) = h_2(t)$ to a desired value H defined by:

$$\begin{aligned} &\text{for t} \in [0 \quad , 200[\,,\ \text{H=4cm} \\ &\text{for t} \in [200\ , 400[\,,\ \text{H=6cm} \end{aligned} \quad \textbf{(68)}$$

The parameter values of the coupled tanks process are presented in the following table:

Table 2. Values of the parameters of the coupled tanks **[22]**

Area of the coupling orifice: C	208.2 cm^2
Area of the outlet orifice: c_{12}	0.58 cm^2
Cross-section area of Tank 1 to Tank 2: c_2	0.24 cm^2
Gravitational constant: g	981 cm/s^2

To obtain realistic results, the simulation is carried out using the following input:

$$0 \text{ cm}^3/s \le u \le 50 \text{ cm}^3/s \quad \textbf{(69)}$$

The simulation results are shown in **Figs 9-11.** We notice from the tracking curve **Fig.9** and the tracking error curve **Fig. 10** that the output follows the reference signal

H. The evolution of the sliding surface trajectory is shown in **Fig. 11**. We notice the convergence to zero of the system which proves that the attractiveness of the sliding surface is guaranteed.

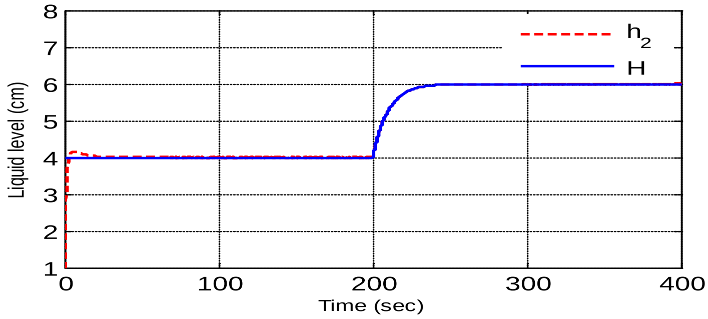

Fig. 9. The trajectories of liquid level h_2 and desired liquid level H.

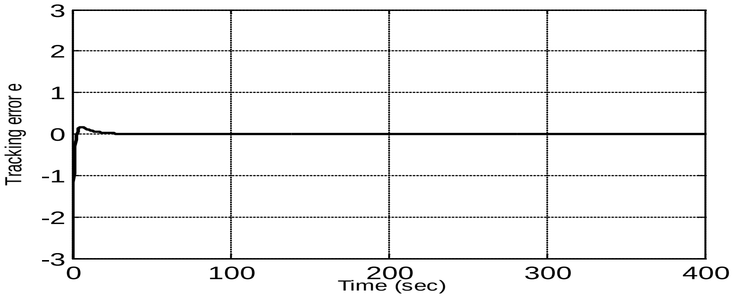

Fig. 10. Evolution of the tracking error.

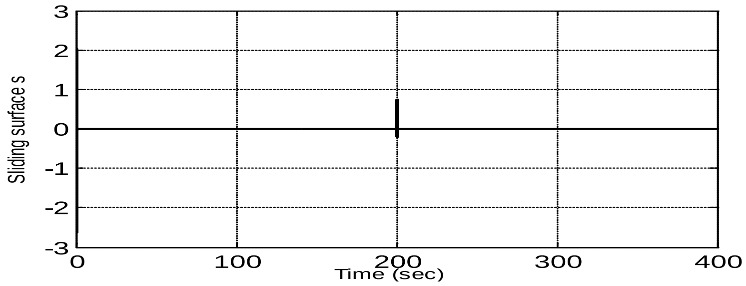

Fig. 11. Evolution of the classic sliding surface s.

7. Conclusion

In this paper, adaptive sliding mode controller is developed for a class of nonlinear single input single output disturbed systems.

In the first part, we proposed a solution to reduce the chattering phenomenon. In fact, an adaptive proportional integral controller is used to approximate the discontinuity term of the classical sliding mode control. In the second part, we proposed another solution to avoid the rapprochement phase of the sliding surface. This solution includes an adaptive proportional derivative controller and an integral surface. Finally, based on the Lyapunov stability approach, we demonstrate that the proposed adaptive sliding mode control scheme can guarantee the global stability and the robustness of the closed loop system with respect to disturbance. The simulation results of the inverted pendulum and the coupled tanks system have shown the effectiveness of the proposed control method and the good performances in comparison with other recent SMC methods proposed in the literature.

Acknowledgement:

We thank the ministry of higher education and scientific research of Tunisia for funding this work.

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