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Nonstationary sinusoidal unknown inputs multiobserver for discrete-time nonlinear systems

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Abstract. In this work, a nonstationary sinusoidal unknown inputs multiobserver is proposed for discrete-time nonlinear systems represented by a decoupled multimodel. The existence conditions of this type of multiobserver will be addressed and a new strategy allowing the minimization of the impact of nonstationary sinusoidal unknown inputs on the state estimation error will be developed. The motivation of this methodology resides in the fact that it provides a simultaneous and accurate states and nonstationary sinusoidal unknown inputs estimation as well as a robust estimation error.

Keywords: Discrete-time, Nonlinear systems, Decoupled multimodel, Sinusoidal unknown inputs, State and unknown inputs estimation, Multiobserver.

1 INTRODUCTION

Generally, the dynamical behavior of a system is described by the evolution of the state variables. The synthesis of control laws, the fault diagnosis or system monitoring necessitate the knowledge of all state variables [6, 8, 15–17]. However, for technical, economical, and practical reasons the availability of all states can be a difficult task. To solve this problem, researchers tend to estimate the states using observers. On the other hand, it is well known that the system can be affected by unmeasurable inputs on its states and outputs [5, 21]. The presence of these unknown inputs have adverse effects on the system behavior's and can even make the state estimation much more difficult. The state estimation will thus be biased and will never be perfectly accomplished under these conditions since the unknown inputs intervene in the system dynamics and sometimes in a nonlinear way. So, it is necessary to design an observer that is able to reconstruct the states and the unknown inputs from available measurements despite the presence of unknown inputs. This observer undergoes additional structural

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conditions compared to the classical Luenberger observers and provides an accurate and robust estimation. Furthermore, this observer leads to an uncoupled state estimation error compared with the unknown inputs due to auxiliary degrees of freedom.

Two varieties of the unknown input observer are frequently encountered. The first one serves to simultaneously estimate the states and the unknown inputs [1, 5, 9, 19, 20]. This approach assumes a priori some knowledge on these unmeasurable inputs and takes into account the estimation error, while the second one seeks to perfectly uncouple the state estimation error from these inputs, via matrix transformation, without any knowledge about the unknown inputs [3, 13, 25, 26]. In these circumstances, the estimates of the unknown inputs are taken into account only after the estimation of the system's state. The state estimation is used to estimate the unknown inputs whose the quality will be directly related to the quality of the state estimation.

In the literature, the unknown input observers were developed for linear systems. Actually, there exist many techniques for the state estimation of the linear systems with unknown inputs[2, 5, 10–12]. However, most physical systems have nonlinear behaviors or a complex structure. So, the linearization approach is no longer efficient to accurately describe the system behavior. In the framework of the unknown input observers synthesis, several works have been developed for specific classes of nonlinear systems [3, 13, 18]. However, the synthesis of observers for this type of systems knew many challenges because of their complexity and the lack of generality. To overcome this problem, the multimodel approach can be a convenient alternative [14, 20].

The motivation of this approach stems from the fact that it is often difficult to design a model which takes account of all the complexities of the studied system. Indeed, the multimodel approach constitutes a very interesting mathematical representation for nonlinear systems because it enables the representation of any nonlinear behavior, whatever its degree of complexity, by a simple structure based on linear partial models interpolated by weighting functions. They have a simple structure making them mathematically easier, more exploitable and thus allowing the extension of some linear results to the nonlinear systems[21, 23]. The extensions of unknown input observers to nonlinear systems represented by a Takagi-Sugeno multimodel structure are investigated in [8, 9, 19, 25, 26].

However, this structure does not take into account the structural changes. Indeed, it has an invariable number of states regardless of the complexity of the system behavior in the different operating zones. Thus, the dimension of the partial models is imposed by the partial model of greatest dimension. However, the dimensions of partial models can be reduced by adding other operating zones in order to reduce the complexity of the system in each zone. Then, the obtained multimodel can be over-parameterized and its complexity uselessly increase.

For these reasons, we were interested in exploiting the decoupled multimodel. This form of multimodel respects the degree of complexity in each operating zone. This leads to the reduction of the number of parameters to be identified. So, we can clearly see the significance of this multimodel structure. In this

work, we designed a nonstationary sinusoidal unknown inputs multiobserver to estimate the states and the unknown inputs for complex nonlinear systems simultaneously. The proposed multiobserver uses integrators to ensure a good state estimation in the presence of nonstationary sinusoidal unknown inputs.
The present paper is organized as follows. In the second part, the decoupled multimodel representation and the multiobserver are presented. The third part provides the convergence conditions and the multiobserver synthesis. In the fourth part, simulation example is proposed to illustrate the significance of the proposed multiobserver. A conclusion finishes this paper.

2 Problem statement and preliminaries

2.1 Structure of the decoupled multimodel with unknown inputs

The multimodel representation of a nonlinear system have different structures. The various partial models are interconnected in order to generate the global output of the system.
In this context, we can enumerate two main structures of multimodel. The first is the Takagi-Sugeno multimodel [24] where partial models are homogeneous (sharing the same state space). The second is the decoupled multimodel, their partial models are heterogenous (different dimension). This structure was firstly proposed by Filev [4].
A discrete-time non linear system can be described by the following decoupled multimodel:

$$\begin{cases} x_i(k+1) = A_i x_i(k) + B_i u(k) + E_{e_i} P_s(k) \\ y_i(k) = C_i x_i(k) \\ y_{MM}(k) = \sum_{i=1}^{Nm} \mu_i(\nu(k)) y_i(k) + E_s P_s(k) \end{cases} \tag{1}$$

where:
$x_i(k) \in \Re^{n_i}$ and $y_i(k) \in \Re^p$ are the state and the output vectors of the i^{th} partial model,respectively.
$y_{MM}(k) \in \Re^p$ and $u(k) \in \Re^m$ denote the multimodel output and the known input vectors, respectively.
$P_s(k) \in \Re^l$ is the nonstationary sinusoidal unknown inputs characterized by its amplitude P_m and its period T_s (the multiple integer of the sampling period).
The matrices $A_i \in \Re^{n_i \times n_i},\ B_i \in \Re^{n_i \times m},\ C_i \in \Re^{p \times n_i},\ E_{e_i} \in \Re^{n_i \times l}\ and\ E_s \in \Re^{p \times l}$are known and appropriately dimensioned.
Nm is the number of partial models.

$\mu_i(\nu(k))$ are the weighting functions. They depend on decision variables which can be measurable or unmeasurable.
They are generally chosen in order to verify the following properties:

$$\begin{cases} \sum_{i=1}^{Nm} \mu_i(\nu(k)) = 1 \\ 0 \le \mu_i(\nu(k)) \le 1 \end{cases} \forall\ i = 1, \ldots, Nm, \forall\ k \tag{2}$$

These functions are constructed in several ways to ensure the transition between different partial models.

$$\mu_i(\nu(k)) = \frac{e^{\left(\frac{-(\nu(k)-c_i)^2}{\sigma_d{}^2}\right)}}{\sum_{i=1}^{Nm} e^{\left(\frac{-(\nu(k)-c_i)^2}{\sigma_d{}^2}\right)}}, \qquad i = 1, 2, \ldots, Nm \tag{3}$$

where:
c_i $(i = 1, \ldots, Nm)$ are the centers and σ_d is the dispersion.
$\nu(k)$ is the decision variable. These variables can be measurable (input or output signals) or measurable (state variables).
Thereafter, in order to obtain a compact multimodel structure, the vector $x_{cf}(k)$ can be defined as follows:

$$\begin{aligned} x_{cf}(k) &= \begin{bmatrix} x_1^T(k) & \cdots & x_i^T(k) & \cdots & x_{Nm}^T(k) \end{bmatrix}^T \in \Re^n, \\ n &= \sum_{i=1}^{Nm} n_i \end{aligned} \tag{4}$$

The decoupled multimodel (1) can be written as follow:

$$\begin{cases} x_{cf}(k+1) = A_{cf}x_{cf}(k) + B_{cf}u(k) + E_{e_{cf}}P_s(k) \\ y_{MM}(k) = C_{cf}(k)x_{cf}(k) + E_sP_s(k) \end{cases} \tag{5}$$

where:

$$A_{cf} = \begin{bmatrix} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & A_i & & \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & A_{Nm} \end{bmatrix} \in \Re^{n\times n} \; B_{cf} = \begin{bmatrix} B_1{}^T & \cdots & B_i{}^T & \cdots & B_{Nm}{}^T \end{bmatrix}^T \in \Re^{n\times m}$$

$$C_{cf}(k) = \begin{bmatrix} \mu_1(\nu(k))\,C_1 & \cdots & \mu_i(\nu(k))\,C_i & \cdots & \mu_{Nm}(\nu(k))\,C_{Nm} \end{bmatrix}$$
$$C_{cf}(k) \in \Re^{p\times n}$$

$$E_{e_{cf}} = \begin{bmatrix} E_{e_1}{}^T & \cdots & E_{e_i}{}^T & \cdots & E_{e_{Nm}}{}^T \end{bmatrix}^T \in \Re^{n\times l}$$

Exploiting the properties of convex sum and considering the bloc matrix of the form:

$$\tilde{C}_{cf_i} = \begin{bmatrix} 0 \cdots & C_i & \cdots 0 \end{bmatrix} \in \Re^{p\times n}$$

The matrix $C_{cf}(k)$ can be rewritten in the following form:

$$C_{cf}(k) = \sum_{i=1}^{Nm} \mu_i(\nu(k))\tilde{C}_{cf_i} \tag{6}$$

2.2 Multiobserver structure

To provide a simultaneous estimation of the state and the sinusoidal unknown inputs, we consider the proportional multiobserver (7) coupled to the Q-integral observer described by (8) . The reconstruction of the state variables of the multimodel with sinusoidal unknown inputs (1) is performed by synthesizing a multiobserver of the following form [7, 20]:

$$\begin{cases} \hat{x}_i(k+1) = A_i\hat{x}_i(k) + B_iu(k) + E_{e_i}\hat{P_{s_0}}(k) \\ \qquad\qquad\qquad\qquad + K_i\left[y(k) - \hat{y}_{MM}(k)\right] \\ \hat{y}_i(k) = C_i\hat{x}_i(k) \\ \hat{y}_{MM}(k) = \sum_{i=1}^{N_m} \mu_i(\nu_i(k))\hat{y}_i(k) + E_s\hat{P}_{s_0}(k) \end{cases} \tag{7}$$

However, the sinusoidal unknown inputs estimation is made via the following observer:

$$\begin{cases} \hat{P}_{s_q}(k+1) = \hat{P}_{s_q}(k) + K_q\left[y(k) - \hat{y}_{MM}(k)\right] + \hat{P}_{s_{q+1}}(k), \\ q \in [0,\, Q-1] \\ \hat{P}_{s_Q}(k+1) = \hat{P}_{s_Q}(k) + K_Q\left[y(k) - \hat{y}_{MM}(k)\right] \end{cases} \tag{8}$$

where:
$\hat{x}_i(k) \in \Re^{n_i}$ is the state estimates of the i^{th} partial model.
$\hat{y}_i(k), \hat{y}_{MM}(k), y(k) \in \Re^p$ denote the estimates of $y_i(k)$, $y_{MM}(k)$ and the measured output, respectively.
$\hat{P}_{s_0}(k) \in \Re^l$ is the sinusoidal unknown input estimation.
$K_i \in \Re^{n_i \times p}$ represents the gain of the i^{th} partial model.
$K_q \in \Re^{l \times p}$ is the gain of q-integral observer ($q \in [0,\, Q]$).
In order to rewrite the multiobserver (7) in a compact form, we define the following vector:

$$\hat{x}_{cf}(k) = \left[\, \hat{x}_1^T(k) \cdots \hat{x}_i^T(k) \cdots \hat{x}_{Nm}^T(k) \,\right]^T \in \Re^n \tag{9}$$

The multiobserver is then:

$$\begin{cases} \hat{x}_{cf}(k+1) = A_{cf}\hat{x}_{cf}(k) + B_{cf}u(k) + E_{e_{cf}}\hat{P}_{s_0}(k) \\ \qquad\qquad\qquad\qquad + K_{p_{cf}}\left[y(k) - \hat{y}_{MM}(k)\right] \\ \hat{y}_{MM}(k) = C_{cf}(k)\hat{\mathrm{x}}_{cf}(k) + E_s\hat{P}_{s_0}(k) \end{cases} \tag{10}$$

where $\hat{x}_{cf}(k) \in \Re^n$ is the estimated of $x_{cf}(k)$ and $K_{p_{cf}} \in \Re^{l \times n}$ is the gain of the global multiobserver.

2.3 Definitions of the estimation errors

In this section, we establish the expressions of the estimation errors of states and the sinusoidal unknown inputs in order to study their convergences.

The difference operator will be introduced in the expression of error.
The q^{th}-difference operator of a signal $P_s(k)$ is defined as follow:

$$\Delta^{(q)} P_s(k) = \Delta^{(q-1)}(\Delta P_s(k)) = \Delta(\Delta^{(q-1)} P_s(k))$$

Let us define the state and the sinusoidal unknown inputs estimation errors:

$$\begin{cases} e_x(k) = x(k) - \hat{x}(k) \\ e_{p_q}(k) = \Delta^{(q)} P_s(k) - \hat{P}_{s_q}(k), q = 0 \dots Q \end{cases} \tag{11}$$

where: $\Delta^{(0)} P_s(k) = P_s(k)$.
Starting from this definitions and using the expression of $x_{cf}(k)$, $\hat{x}_{cf}(k)$ and $e_x(k)$ given by the equations (5), (10) and (11), respectively, the generatrix function of the state estimation error is given by the following expression:

$$e_x(k+1) = [A_{cf} - K_{p_{cf}} C_{cf}(k)] e_x(k) + [E_{e_{cf}} - K_{p_{cf}} E_s] e_{p_0}(k) \tag{12}$$

Using (8) and (11), the dynamics of the sinusoidal unknown inputs estimation is written:

$$e_{p_q}(k) = -K_q C_{cf}(k) e_x(k) - k_q E_s e_{p_0}(k) + e_{p_q}(k) + e_{p_{q+1}}(k) \tag{13}$$

$$\begin{aligned} e_{p_Q}(k+1) = -K_Q C_{cf}(k) e_x(k) - k_Q E_s e_{p_0}(k) + e_{p_Q}(k) \\ + \Delta^{(Q+1)} P_s(k) \end{aligned} \tag{14}$$

The augmented error vector is defined as follow:

$$\begin{aligned} \psi(k) = \begin{bmatrix} e_x^T(k) \; e_{p_0}^T(k) \cdots e_{p_q}^T(k) \cdots e_{p_Q}^T(k) \end{bmatrix}^T \in \Re^{\vartheta}, \\ \vartheta = n + l(q+1), \end{aligned} \tag{15}$$

This vector allows to write the dynamics of the state estimation error, the sinusoidal unknown input estimation error and their differences in a compact form as follows:

$$\psi(k+1) = [\Omega - K_r \theta(k)] \psi(k) + \Pi \Delta^{(Q+1)} P_s(k) \tag{16}$$

where:

$$\Omega = \begin{bmatrix} A_{cf} & E_{e_{cf}} & 0 & \cdots & & 0 \\ 0 & I_l & I_l & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ & & & \ddots & I_l & I_l \\ 0 & & \cdots & & 0 & I_l \end{bmatrix} \in \Re^{\vartheta \times \vartheta}$$

$$K_r = \begin{bmatrix} K_{p_{cf}}^T \; K_0^T \cdots K_Q^T \end{bmatrix} \in \Re^{\vartheta \times p}$$

$$\theta(k) = \begin{bmatrix} C_{cf}(k) \; E_s \; 0 \cdots 0 \end{bmatrix} \in \Re^{p \times \vartheta}$$

$\Pi = \begin{bmatrix} 0 \cdots 0 \; I_l \end{bmatrix}^T \in \Re^{\vartheta \times l}$

Taking into consideration the properties of the convex sum and the expression described by (6), we can write:

$$\varphi(k) = \sum_{i=1}^{Nm} \mu_i(\nu(k)) \left[\Omega - K_r \theta_i\right]$$
$$\theta_i = \left[\, \tilde{C}_{i_{cf}} \; E_s \; 0 \cdots 0 \right]$$

Using this expression, (16) becomes:

$$\psi(k+1) = \varphi(k)\psi(k) + \Pi \Delta^{(Q+1)} P_s(k) \tag{17}$$

Thereafter, our aim is to adjust the gain K_r which ensures exponential convergence of the estimation error towards zero. This gain should provide errors estimation decoupled from sinusoidal unknown inputs, so it should minimize the impact of sinusoidal unknown inputs on the state estimation error.

3 Main results

3.1 Convergence conditions of estimation errors

The reader can notice that equation (17) shows clearly that the estimation errors are directly affected by the $(Q+1)^{th}$ difference of the sinusoidal unknown inputs.

For polynomial unknown inputs with q known degree, the perfect decoupling can be achieved by involving $(q+1)$ integral actions [20, 22]. However, for nonstationary sinusoidal unknown inputs, convergence conditions are not yet treated.

In this section, we develop a particular study to provide the convergence conditions the estimation errors and to ensure a good decoupling of these estimation errors from nonstationary sinusoidal unknown inputs.

Consider the sinusoidal signal defined by the following expression:

$$P_s(k) = P_m \sin(\frac{2k\pi}{T_s} + \varphi_s) \tag{18}$$

where P_m and T_s are the amplitude and the period of the sinusoidal unknown input, respectively.

Taking everything into account, the expression of q^{th} difference is expressed as follows:

✓ For peer values of q:

$$\Delta^{(q)} P_s(k) = (-1)^{(\frac{q}{2})} \left[2 \sin(\frac{\pi}{T_s})\right]^q P_m \sin(\frac{2k\pi}{T_s} + \frac{q\pi}{T_s} + \varphi_s) \tag{19}$$

✓ For odd values of q:

$$\Delta^{(q)} P_s(k) = (-1)^{(\frac{q-1}{2})} \left[2 \sin(\frac{\pi}{T_s})\right]^q P_m \cos(\frac{2k\pi}{T_s} + \frac{q\pi}{T_s} + \varphi_s) \tag{20}$$

The convergence of $\Delta^{(q)} P_s(k)$ to the neighborhood of zero is ensured by satisfying some conditions.
Indeed, the q successive differences of $P_s(k)$ drifts the following term:

$$\varsigma = \left[2\sin(\frac{\pi}{T_s}) \right]^q P_{m,} \tag{21}$$

ς denotes the maximum amplitude of $\Delta^{(q)} P_s(k)$. Thereafter, there exists a period T_{sl} which checks $\left| 2\sin(\frac{\pi}{T_{sl}}) \right| < 1$, such as for $T_s > T_{sl}$, a judicious choice of $q \in \mathbb{N}^*$ guaranteed the convergence of ς, therefor of $\Delta^{(q)} P_s(k)$, towards ε_d infinitely small.

For any sinusoidal unknown input $P_{s_j}(k)$ characterized by its amplitude P_{m_j} and its period T_{s_j} which check:

$$P_{m_j} \leq P_m \tag{22}$$

$$T_{s_j} \geq T_s > T_{sl} \tag{23}$$

it becomes easy to deduce that the absolute value of the maximum amplitude of $\Delta^{(q)} P_{s_j}(k)$ is lower or equal to the maximum amplitude of $\Delta^{(q)} P_s(k)$.
Thus, if we bring back the value of ς towards zero by choosing the value of q suitably, we can consider that $\Delta^{(q)} P_s(k)$ and $\Delta^{(q)} P_{s_j}(k)$ converge towards ε_d infinitely small.
The q value tending the q^{th} difference of the nonstationary sinusoidal unknown input to the neighborhood of zero can be determined by the following rule:

$$q = f_q\left(\frac{\log(\varepsilon_d) - \log(P_m)}{\log(2\sin(\frac{\pi}{T_s}))}\right) \tag{24}$$

where f_q is a function defined by:

$$\begin{aligned} f_q :& \mathbb{R} \to \mathbb{Z} \\ & r \to z \end{aligned}$$

It rounds the elements of r to z which is the nearest integers greater than or equal to r.

The obtained value of q ensures the convergence of $\Delta^{(q)} P_s(k)$, for all $T_{s_j} \geq T_s > T_{sl}$ and $P_{m_j} \leq P_m$, towards ε_j infinitely small ($\varepsilon_j \leq \varepsilon_d$).
The estimation errors will be then relatively decoupled from the sinusoidal unknown inputs.
By considering this results, the estimation errors (17) become:

$$\begin{aligned} \psi(k+1) &= \varphi(k)\psi(k) \\ &= [\Omega - K_r\theta(k)]\,\psi(k) \end{aligned} \tag{25}$$

$$\varphi(k) = \sum_{i=1}^{Nm} \mu_i(\nu(k))\,[\Omega - K_r\theta_i] \tag{26}$$

3.2 Multiobserver design

The convergence of the estimation error is depends on the stability of system (25).
If the matrix ($\varphi(k) = \Omega - K_r\theta(k)$) is stable , this error converges asymptotically or exponentially to zero. Thus, the aim is to determine the gain K_r that leads to a robust estimation error against the sinusoidal unknown inputs.
However, the search for a solution to the equation (25) governing the dynamics of the estimation error proves to be difficult taking into account the variations of the matrix $\varphi(k)$.
In order to surmount these difficulties, we used the second method of Lyapunov. This concept, also called α-stability of the observer, enables the study of the exponential convergence of the estimation error. It allows to impose dynamic performances on the observer by guaranteeing a certain convergence speed of the estimation error to the final value.
Let us define the Lyapunov functional. According to the Lyapunov theory, the exponential stability is guaranteed if there is a Lyapunov functional $V(k)$ ¿ 0 such as:

$$\exists\ X = X^T > 0,\ \alpha > 0 : \Delta V(k) + 2\ \alpha\ V(k) < 0 \tag{27}$$

Where:
α is the decay rate which serves to quantify the convergence rate of the estimation error.
$\Delta V(k) = V(k+1) - V(k)$ is the variation of the Lyapunov functional.
The quadratic Lyapunov functional that provides the convergence conditions of the estimation error towards zero, is expressed as follows:

$$V(k) = \psi(k)^T\ X\ \psi(k),\ X = X^T > 0 \tag{28}$$

Taking into account equations (25) and (28), the variation of the Lyapunov functional becomes:

$$\Delta V(k) = \psi(k)^T\{(\Omega - K_r\theta(k))^T X(\Omega - K_r\theta(k)) - X\}\psi(k) \tag{29}$$

The inequality (27) can be rewritten as follows:

$$\psi(k)^T\{(\Omega - K_r\theta(k))^T X(\Omega - K_r\theta(k)) + (2\alpha - 1)X\}\psi(k) < 0 \tag{30}$$

To ensure the stability condition, it suffices to check the following LMI(s):

$$(\Omega - K_r\theta(k))^T X(\Omega - K_r\theta(k)) + (2\alpha - 1)X < 0 \tag{31}$$

By replacing equations (26) in (31) and considering the properties of weigting functions, we obtain:

$$(\Omega - K_r\theta_i)^T X(\Omega - K_r\theta_i) + (2\alpha - 1)X < 0 \tag{32}$$

Assuming that:

$$W = X\ K_r \tag{33}$$

We can therefore write:

$$(1-2\alpha)X-(X\Omega-W\theta_i)^TX^{-1}(X\Omega-W\theta_i)>0 \tag{34}$$

Applying Schur complement, inequality (34) can then be written as follows:

$$\begin{pmatrix}(1-2\alpha)X & (X\Omega-W\theta_i)^T\\ (X\Omega-W\theta_i) & X\end{pmatrix}>0,\, i=1...N_m \tag{35}$$

If this inequality has a solution for a rate given decay,

$$0<\alpha<0.5,$$

the multiobserver gain is determined by:

$$K_r=X^{-1}\ W \tag{36}$$

4 Simulation results

Consider the discrete-time nonlinear system described by a decoupled multi-model (1), defined by two partial models with different dimensions.
The numerical values of the matrices of the partial models are given as follows:
Partial model 1:

$$A_1=\begin{bmatrix}-0.32 & -0.04\\ 1 & 0\end{bmatrix},\ B_1=\begin{bmatrix}1\\0\end{bmatrix};$$
$$C_1=\begin{bmatrix}0\ 0.3\end{bmatrix},\quad E_{e_1}=\begin{bmatrix}0.2\\0.1\end{bmatrix}.$$

Partial model 2:

$$A_2=\begin{bmatrix}-0.15 & 0.45 & 0.3\\ -0.1 & -0.7 & -0.2\\ -0.2 & 0.3 & -0.8\end{bmatrix},\ B_2=\begin{bmatrix}0.5\\0.2\\0.1\end{bmatrix};$$
$$C_2=\begin{bmatrix}1\ 0\ 0\end{bmatrix},E_{e_2}=\begin{bmatrix}0.1\\0.2\\0.3\end{bmatrix}$$
$$E_s=0.4$$

The decision variable $\nu(k)$ is the input signal $u(k)\in[0,1]$. Dispertion and centers have the following values:

$$\sigma_d=0.2, c_1=0.25 \text{ and } c_2=0.75$$

Figure 1 shows the evolutions of the input:

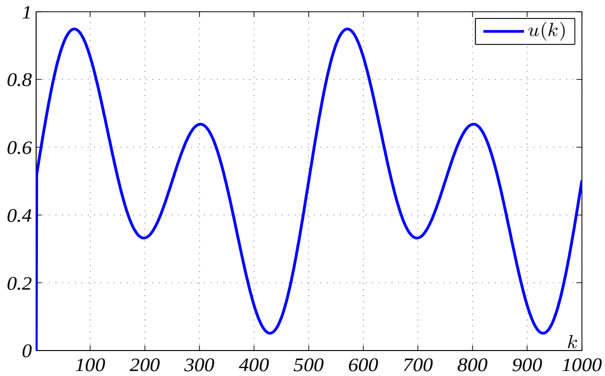

Fig. 1. Evolution of the input.

The evolution of the real nonstationary sinusoidal unknown input is shown in figure 2:

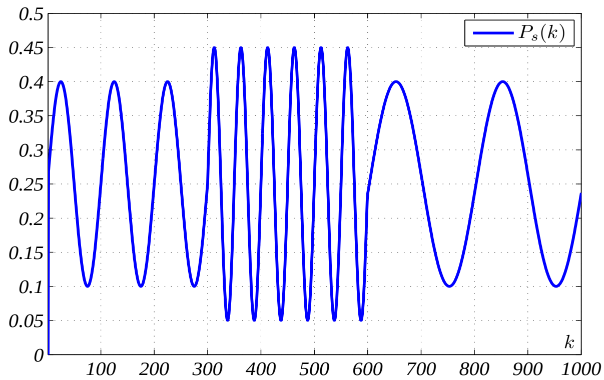

Fig. 2. Evolution of the real nonstationary sinusoidal unknown input.

The nonstationary sinusoidal unknown input is characterized by its maximum amplitude $P_m = 0.2$ and its minimum period $T_s = 50$ (see figure 2).
A good decoupling can be accomplished through realising integrators which make possible to minimize the effect of the sinusoidal unknown input on the estimation errors.
By fixing, for example, $\varepsilon_d = 10^{-3}$ the number of integral actions, determined starting from the equation (24), is equal to 3.

The evolution of the 3^{rd} difference of $P_s(k)$ is shown in figure 3:

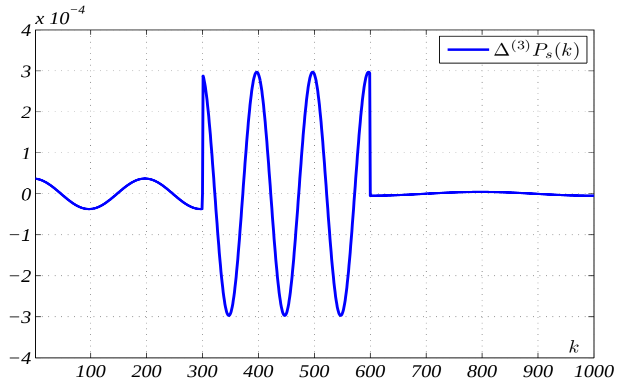

Fig. 3. Evolution of $\Delta^{(3)}P_s(k)$.

From figure 3, we can clearly see that 3 integral actions can tend $\Delta^{(3)}P_s(k)$ towards $\varepsilon_0 < \varepsilon_d$.
By considering the decay rate $\alpha = 0.05$ and basing on LMI(s) conditions of section (III-B), the obtained gain is:

$$k_r = \begin{bmatrix} 0.2999\ 0.2932\ 0.1772\ 0.2438 \\ 0.3043\ 2.9585\ 1.5583\ 0.3208 \end{bmatrix}^T \tag{37}$$

Figures 4, 5, 6, 7 and 8 show the evolutions of the states, their estimates and the estimation errors.

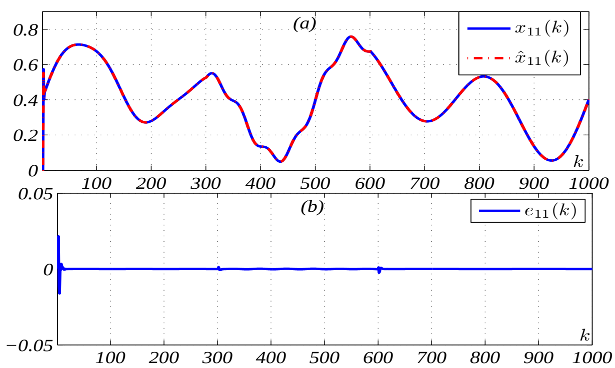

Fig. 4. Partial model 1: (a) Evolutions of the state 1 and its estimate (b) Evolution of the estimation error.

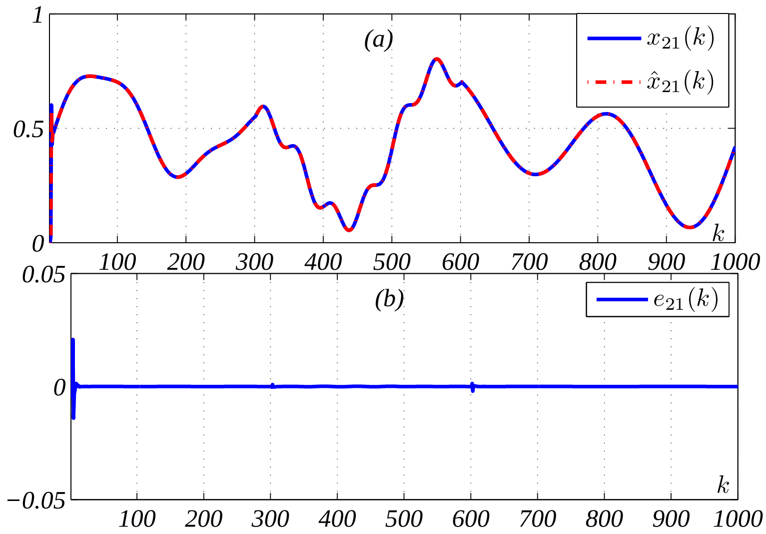

Fig. 5. Partial model 1: (a) Evolutions of the state 2 and its estimate (b) Evolution of the estimation error.

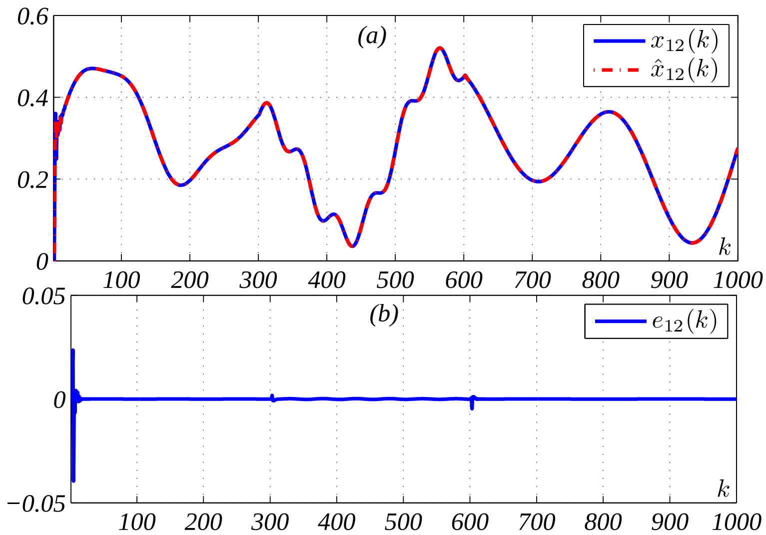

Fig. 6. Partial model 2: (a) Evolutions of the state 1 and its estimate (b) Evolution of the estimation error.

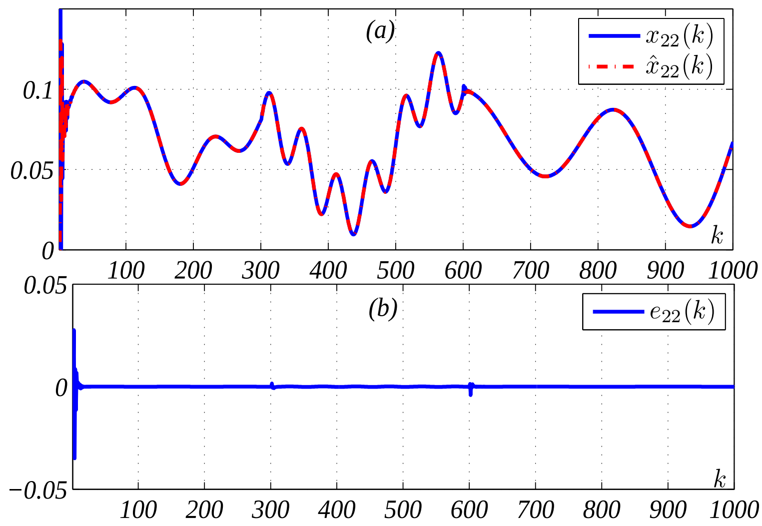

Fig. 7. Partial model 2: (a) Evolutions of the state 2 and its estimate (b) Evolution of the estimation error.

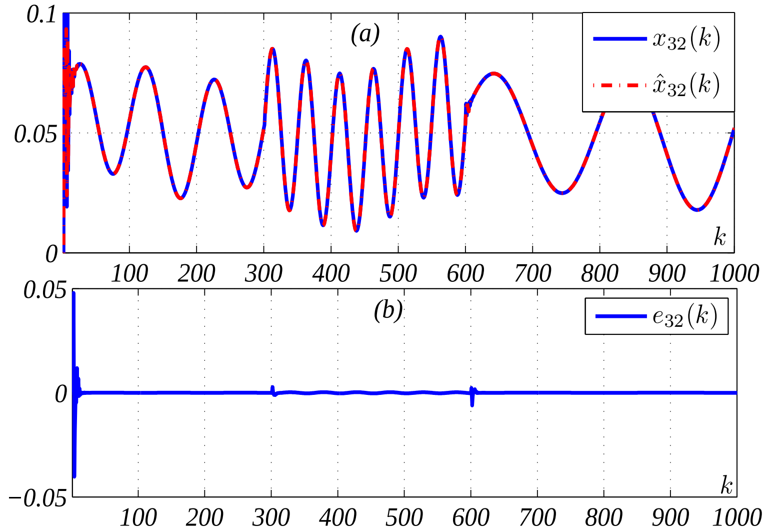

Fig. 8. Partial model 2: (a) Evolutions of the state 3 and its estimate (b) Evolution of the estimation error.

These figures show a good estimation because the states errors estimation can be considered negligible. Indeed, it is clearly seen that the components of

the estimated states vectors by the multiobserver are coincided with those of the partial models.
The nonstationary sinusoidal unknown input, its estimated and estimation error are plotted in figure 9. This figure lets appear a good quality of unknown input estimation. It shows that the estimation error remain to the neighbor of zero after a short transitional phase.

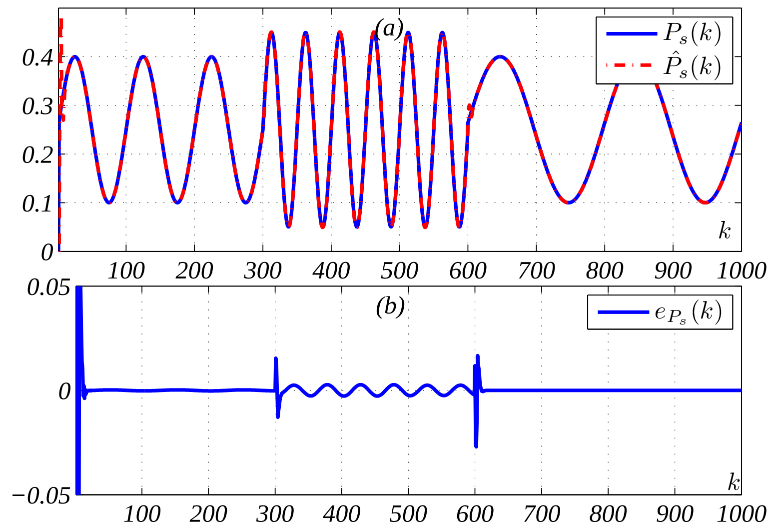

Fig. 9. (a) Evolutions of the nonstationnary sinusoidal unknown input and its estimate (b) Evolution of the estimation error.

5 Conclusion

In this paper, we designed and studied the exponential stability of nonstationary sinusoidal unknown inputs multiobserver. Stabilities and convergence conditions were formulated using linear matrix inequalities which were established by the use of quadratic Lyapunov functions.
Indeed, we correctly estimated the states and sinusoidal unknown inputs simultaneously while ensuring a good compromise between the accuracy of the estimates and the model complexity. Finally, all the results were validated by simulations on an academic example.

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