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Sliding Surface Controller Design Based On the LMIs with Application to a Two Links Robot

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Abstract. In sliding mode approach, the synthesis of the controller is based on the selection of the so called sliding surface. This paper concerned in the first, the design in an optimal way of this surface by using the Linear Matrix Inequality (LMIs) optimization technique. In the second, we propose a sliding mode controller based on the selected surface with application for controlling the position of the tip of the robot with two links. The control action consists of the so called equivalent control and robust control components to ensure the tracking error to zero with no chattering problems. The simulation results show the best performance when the derived controller based on the proposed sliding surface is applied.

Key word: Linear Matrix Inequality, Sliding Mode Control, Two links Robots.

1 Introduction

This design of the sliding surface is the main step in the sliding mode approach [1-5], [9-10]. In many continuous cases, the choice of this surface is achieved by choosing the coefficients of the switching function so that the associated characteristic equation has roots in the open left-half complex plane. Recently; for the control and optimization problem, the LMIs has been accepted as the powerful computation tool [6, 7]. In the control community, it is regarded as a practical solution by transferring the control design problems into LMIs if analytical solutions do not exist or are too difficult to find. LMIs have been used previously for sliding mode control law design [12], [13]. In [12], LMIs is used to synthesis the gains of a sliding mode observer. Sliding surface is then set to be the difference between the observer and the system states, which can cause loss of robustness. In [13], the objective is the design of sliding surface so that the sliding mode dynamics is invariant to matched and mismatched uncertainties. However, the switching surface design is valid for a class of uncertain systems under many assumptions.

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In this paper, the selection of the sliding surface is directly based on the LMIs optimization technique which allows obtaining the optimal coefficients of the switching function. Based on the selected surface in an optimal way, this study propose in the second step, a robust sliding mode controller to steer the switching function to zero in finite time and then, to force the tracking errors to zero in finite time for the tip position of a two links robot [8], [14] in the presence of both matched and unmatched parameters uncertainty. Sliding mode using discontinuous feedback controllers can be used to achieve robust asymptotic output tracking [3]. However, for experimentation, the fast dynamics in the control loop which were neglected in the system model are often excited by the fast switching of the discontinuous term causing the so called "chattering" [11]. A class of techniques to eliminate this phenomenon is based on the use of an observer. However, state observer can cause loss robustness. In this study, the boundary layer solution [9] is used as chattering suppression method and the tracking error tends approximately to zero in finite time.

The main ideas considered in this paper can be presented in two steps. The first considers the problem of selecting a sliding surface for a given system. The LMIs technique is used for the principal reason that is switching function is selected in an optimal way. The second step is to find a sliding mode controller to steer the switching function to zero in finite time. The sliding mode approach is preferred because its robust character to unmodelled dynamics and its insensitivity to external disturbances. The synthesis of the controller is based on the selected sliding surface and uses the Lyapunov function.

The notation used throughout the paper is standard; in particular r denotes the relative degree of the system which is defined to be the least positive integer j for which the derivative $X^{(j)}$, is an explicit function of the input U, such that:

$$\frac{\partial X^{(r)}(t)}{\partial U} \neq 0 \text{ and } \frac{\partial X^{(j)}(t)}{\partial U} = 0 \text{ for } j = 0,1,\ldots,r-1.$$

2 System model of a two links robot

The dynamics of a two links robot control may be expressed as follows [8]:

$$\begin{cases} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = t \\ \dot{t} = Ji - Bt - E\dot{q} \end{cases} \quad (1)$$

where q, $\dot{q}$, $\ddot{q}$, t and i are vectors which represent respectively the position, velocity, acceleration, the torque and current vector applied to the servo motors.

$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ represents the positive-definite symmetric inertia matrix where:

$$M_{11} = I_1 + I_2 + 4m_2 l_1^2 + 4m_2 l_1 l_2 \cos(q_2),\ M_{12} = I_2 + 2m_2 l_1 l_2 \cos(q_2),$$

$$M_{21} = I_2 + 2m_2 l_1 l_2 \cos(q_2) \text{, and } M_{22} = I_2 .$$

With: $q = [q_1\ q_2]^T$: the positions, l_1, l_2: the lengths, m_1, m_2: the masses and I_1, I_2 :the inertias respectively of the first and second segment of the pate.

$C(q, \dot{q}) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ represents the centrifugal forces where:

$$C_{11} = -2m_2 l_1 l_2 \dot{q}_2 \sin(q_2) \text{ , } C_{12} = -2m_2 l_1 l_2 (\dot{q}_1 + \dot{q}_2) \sin(q_2) \text{ ,}$$

$$C_{21} = 2m_2 l_1 l_2 \dot{q}_1 \sin(q_2) \text{ , and } C_{22} = 0 .$$

and the coriolis matrix, $G(q) = \begin{bmatrix} m_2 g l_2 \sin(q_1 + q_2) + m_1 g l_1 \sin(q_1) \\ m_2 g l_2 \sin(q_1 + q_2) \end{bmatrix}$ represents the gravitational forces, with g : the gravitational term.

J, B and E are diagonal matrices representing the thermodynamic parameters and dependent of the temperature and the initial conditions.
In state space, the system model is then as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = f_1(x_1, x_2, x_3, x_4, x_5, x_6) + \\ \qquad g_{11}(x_1, x_4) u_1 + g_{12}(x_1, x_4) u_2 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = f_2(x_1, x_2, x_3, x_4, x_5, x_6) + \\ \qquad g_{21}(x_1, x_4) u_1 + g_{22}(x_1, x_4) u_2 \end{cases} . \qquad (\mathbf{2})$$

Where:

$$q = [x_1\ x_4]^T,\ \dot{q} = [x_2\ x_5]^T,\ \ddot{q} = [x_3\ x_6]^T,\ x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T,\ i = U = [u_1\ u_2]^T ,$$

$$f(x) = [f_1(x)\ f_2(x)]^T =$$

$$-M^{-1}\left\{\left[\dot{M} + C + BM\right]\left[x_3 \quad x_6\right]^T + \left[\dot{C} + BC + E\right]\left[x_2 \quad x_5\right]^T + \left[\dot{G} + BG\right]\right\}$$

$$\text{and } M^{-1}\left[x_1 \quad x_2\right] J = g(x_1, x_4) = \begin{pmatrix} g_{11}(x_1, x_4) & g_{12}(x_1, x_4) \\ g_{21}(x_1, x_4) & g_{22}(x_1, x_4) \end{pmatrix}$$

3 Design of the Sliding Surface by the LMIs Optimization technique

Consider the system model described in the state-space representation with known relative degree ($r \geq 2$):

$$\begin{cases} \dot{X} = f(X, U) \\ y = CX \end{cases} . \tag{3}$$

where X is the state vector, U is the input of the system, y represent the output of the system and C is the matrix of appropriate dimensions.
Let denote the desired trajectory as y_d and the tracking error by $e = y - y_d$. We consider $\dot{e}, e^{(2)}, \ldots, e^{(r-1)}$ the successively derivatives of the tracking error $e(t)$; the local coordinates $Z = [z_1 \; z_2 \; \ldots \; z_{r-1} \; z_r] = [e \; \dot{e} \; e^{(2)} \; \ldots \; e^{(r-1)}]$; and the state variable as $Z_1 = [z_1 \; z_2 \; \ldots \; z_{r-2} \; z_{r-1}]^T$. Then we obtain the system model:

$$\dot{Z}_1 = AZ_1 + Bz_r . \tag{4}$$

Where A and B are the matrices of appropriate dimensions.
The objective is to design an optimal switching surface by using the LMIs optimisation technique such that, when the variables state is steered in this sliding surface the tracking errors are asymptotically forced to zero.
We want to stabilize (4) towards the origin in finite time ($t_f < +\infty$). Then the problem leads back to find a fictive control feedback by the LMIs technique.
Theorem 1: for the system (3), the switching surface can be presented as: $S = z_r - KZ_1$, where $K = LQ^{-1}$ with L and Q the solutions of the feasibility LMIs $QA^T + AQ + L^T B^T + BL < 0$.
Proof
By replacing $z_r = KZ_1$ in (4) we obtain:

$$\dot{Z}_1 = (A + BK)Z_1 . \tag{5}$$

The system (5) is asymptotically stable by the Lyapunov theorem if there exist the symmetric positive definite matrix P such that:

$$(A + BK)^T P + P(A + BK) < 0 .$$

The non linear Lyapunov inequality $(A + BK)^T P + P(A + BK) < 0$ is equivalent to the LMIs $QA^T + AQ + L^T B^T + BL < 0$ such that:
$Q = Q^T = P^{-1}, L = KQ$ and $L^T = QK^T$.

Let define $S = z_r - KZ_1$; the equation $S = 0$ describes the desired dynamics which satisfy the finite time stabilization of vector Z to zero. Then, the optimal switching manifold is defined as $S_0 = \{Z / S(Z,t) = 0\}$ on which system (3) is forced to slide on via the sliding mode controller U. Replacing z_r and $Z_1 = [z_1 \ z_2 \ ... \ z_{r-2} \ z_{r-1}]^T$ by their expressions, the sliding surface is equivalent to:

$S = e^{(r-1)} - K\left[e \ \dot{e} \ e^{(2)} \ ... \ e^{(r-2)}\right]^T$ where K is the solution to the LMIs feasibility.

In case $r = 2$ we have $Z_1 = e, \ z_r = \dot{e}$, $A = 0$ and $B = 1$. The feasibility LMIs in the theorem become: $L^T + L < 0$; $2L < 0$.

In this case the symmetric positive definite matrix $P = Q^{-1}$ correspond to a non null positive constant. Finally, $S = \dot{e} - Ke$, with $K = LQ^{-1}$ is to be chosen for any fixed $L < 0$ and $P = Q^{-1} > 0$.

4 Application for the Control of tip position of a two links robot

We consider in this application, the outputs tracking of a two links robot. First, we formulate the sliding surface by using the proposed technique presented in section III. And then to find a sliding mode control law which steer the switching function to zero in finite time. By considering the system model in (2), the relative degree of the system is $r = 3$ and by using the LMIs optimization technique presented in section III to choose the sliding surface, we obtain:

$$S = \ddot{e} - K[e \ \dot{e}]^T . \quad \textbf{(6)}$$

Where $e = [e_1 \ e_4]^T$ with $e_1 = x_1 - x_{1d}$ and $e_4 = x_4 - x_{4d}$ x_1, x_4 are the controlled outputs and x_{1d} , x_{4d} are the angular desired outputs.

Theorem 2: for the system model defined in (2), the sliding mode controller which makes the output tracking error tend approximately to zero in finite time can be written as follows: $U = Ueq + Ur$, where :

$$Ueq = -g^{-1}\left[f(x) - K\begin{pmatrix} x_2 - x_{2d} \\ x_5 - x_{5d} \\ x_3 - x_{3d} \\ x_6 - x_{6d} \end{pmatrix} - \begin{pmatrix} \dot{x}_{3d} \\ \dot{x}_{6d} \end{pmatrix} \right]$$: is the equivalent control and

$U_r = -m\,g^{-1} sat(S)$: is the robust control term with m is a positive constant.

Proof:

Let consider the candidate Lyapunov function V :

$$V = \frac{1}{2} S^T S \Rightarrow \dot{V} = \frac{1}{2} (\dot{S}^T S + S^T \dot{S}) = S^T \dot{S}$$

By replacing $\dot{S}$, U , U_{eq} by their expressions we obtain $\dot{V} = S^T g U_r$

By assuming $U_r = -mg^{-1} sat(S)$ with m a positive constant and the simple linear saturation function is defined as $\mathrm{sat}(S) = \begin{cases} \mathrm{sign}(S) & for \ \|S\| > e \\ \dfrac{S}{e} & for \ \|S\| \le e \end{cases}$

we obtain:

For any $\varepsilon > 0$, if $\|S\| > e$, $\mathrm{sat}(S) = \mathrm{sign}(S)$, the function $\dot{V} = -mS^T sign(S)$ is negative defined. However, in a small ε-vicinity of the origin, the so called boundary layer [10] ($\|S\| \le e$), $\mathrm{sat}(S) \ne \mathrm{sign}(S)$, $\mathrm{sat}(S) = \dfrac{S}{e}$ is continuous and $\dot{V} = -\dfrac{m}{e} S^T S$.

The system trajectories are confined to a boundary layer of sliding manifold $S = 0$.

5 Simulation results

For a two links robot described by the model (2); the desired angular trajectory are $x_{1d} = \dfrac{\Pi}{2}\cos(t)$ and $x_{4d} = \dfrac{\Pi}{2}\sin(t)$. We consider the numerical parameters of the model as: $g = 9.81$ and for any segment of the pate , the length ($l_1 = 0.11 m$, $l_2 = 0.15 m$) , masse ($m_1 = 0.6 Kg$, $m_2 = 0.4 Kg$), inertia ($I_1 = 0.07$, $I_2 = 0.025$) and the thermodynamics parameters

$$E = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},\ B = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } J = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

The computed LMIs feasibility from the MATLAB returns the optimum values:

$$Q = \begin{bmatrix} 60.4839 & 0 & -20.1613 & 0 \\ 0 & 60.4839 & 0 & -20.1613 \\ -20.1613 & 0 & 60.4839 & 0 \\ 0 & -20.1613 & 0 & 60.4839 \end{bmatrix};$$

$$L = \begin{bmatrix} -60.5172 & 0 & -30.2586 & 0 \\ 0 & -60.5172 & 0 & -30.2586 \end{bmatrix};\ K = \begin{bmatrix} -1.3125 & 0 & -0.9375 & 0 \\ 0 & -1.3125 & 0 & -0.9375 \end{bmatrix}$$

In this example, the constant in the robust control term is: $m = 0.02$, such as $\dot{V}$ is negative as in theorem 2.

For different initials positions without uncertainties and with parameters uncertainties, the figures 1, 6, 11, 16 show the real (-) and the desired (--) trajectory for the Tip position of the robot. Figures 2, 7, 12, 17 and 3, 8, 13, 18 show respectively the angular position of the first link and second link. Figures 4, 9, 14, 19 and 5, 10, 15, 20 show respectively the torque1 and torque2.
The considered parameters uncertainties are the load variations considered as:

$$m_1 = 0.6 + (0.06) randn(t_0, tf) \text{ and } m_2 = 0.4 + (0.04) randn(t_0, tf)$$

with t_0 is the initial time and tf is the final time.

Case 1: Y(0)=Yd(0), X(0)=Xd(0) and without uncertainties

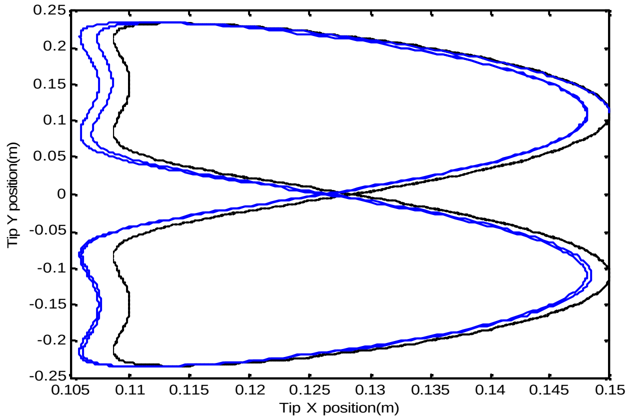

Fig. 1. Tip position of the Robot (_) with the desired trajectory (…)

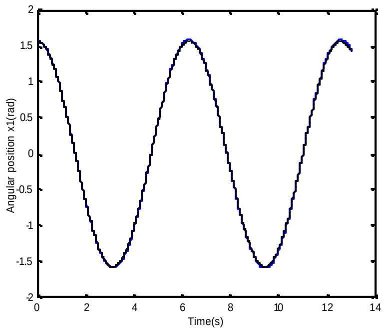

Fig. 2. Angular position, measured (_) and reference(…)(rad)

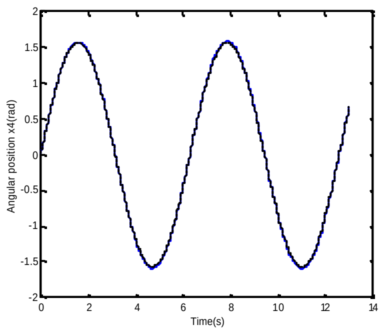

Fig. 3. Angular position, measured (_) and reference(…) (rad)

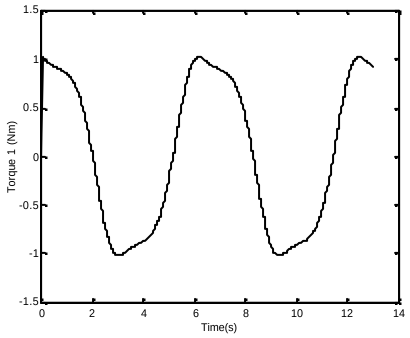

Fig. 4. Joint 1 torque

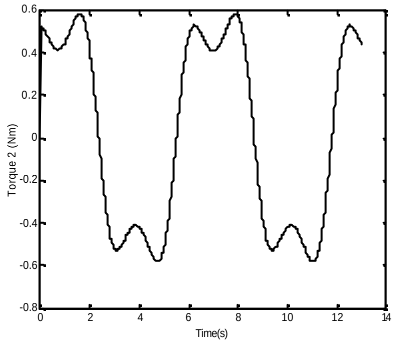

Fig.5. Joint 2 torque

Case 2: Yd(0)- Y(0)= -0.1117 and Xd(0)- X(0)= 0.0911 and without uncertainties

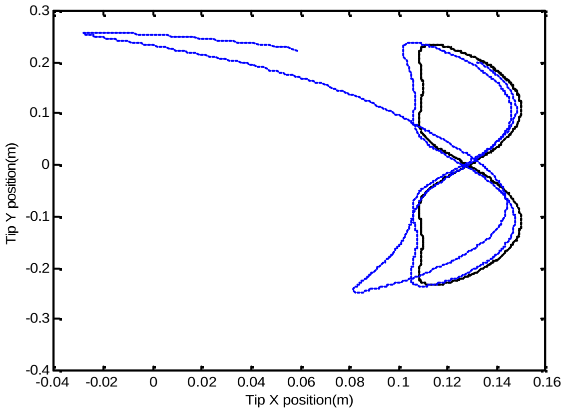

Fig.6. Tip position of the Robot (_) with the desired trajectory (…)

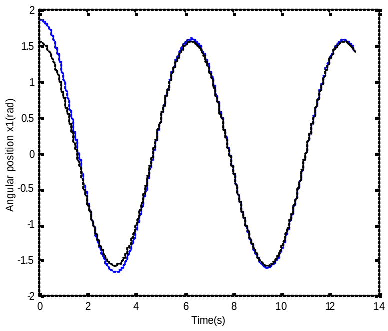

Fig. 7. Angular position, measured (_) and reference(…)(rad)

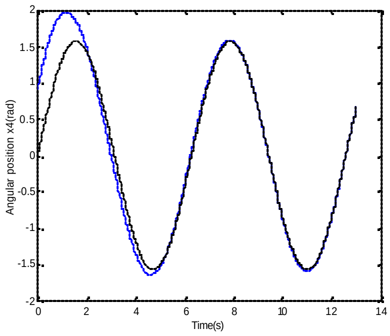

Fig. 8. Angular position, measured (_) and reference(…)(rad)

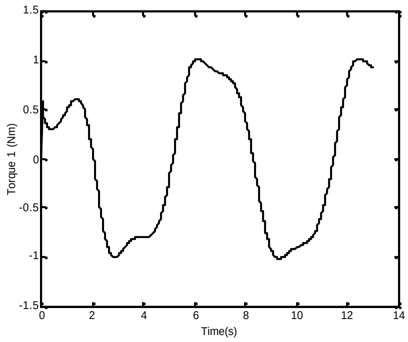

Fig.9. Joint 1 torque

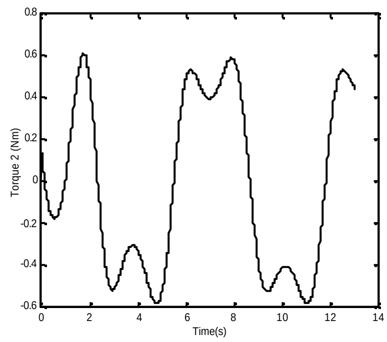

Fig.10. Joint 2 torque

Case 3: Y(0)=Yd(0) and X(0)=Xd(0) with masses variations

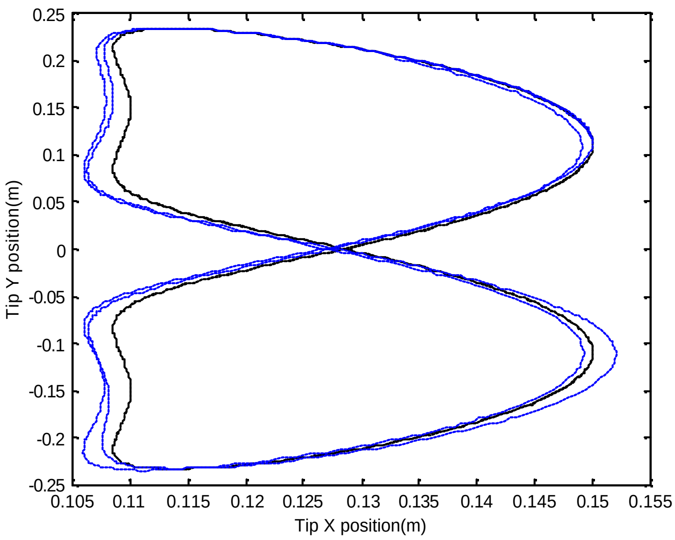

Fig.11. . Tip position of the Robot (_) with the desired trajectory (…)

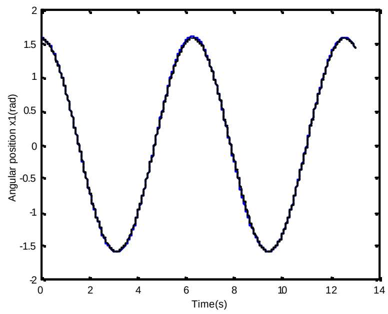

Fig. 12. Angular position, measured (_) and reference(…)(rad)

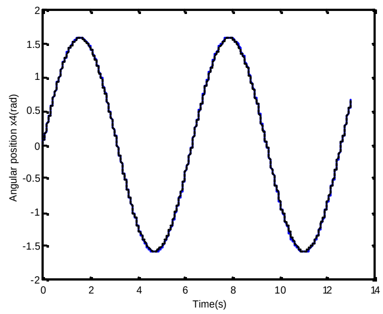

Fig. 13. Angular position, measured (_) and reference(…) (rad)

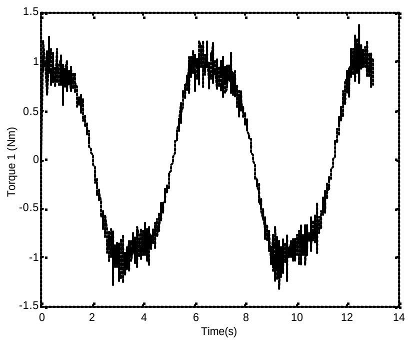

Fig.14. Joint 1 torque

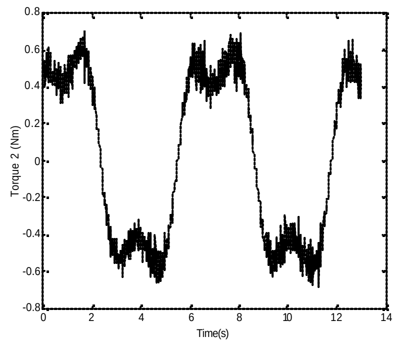

Fig.15. Joint 2 torque

Case 4: Yd(0)- Y(0)= -0.1117 and Xd(0)- X(0)= 0.0911and with masses variations

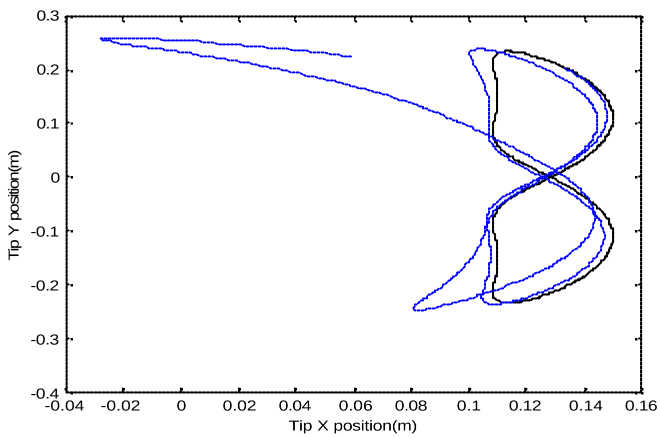

Fig.16. . Tip position of the Robot(_) with the desired trajectory (…)

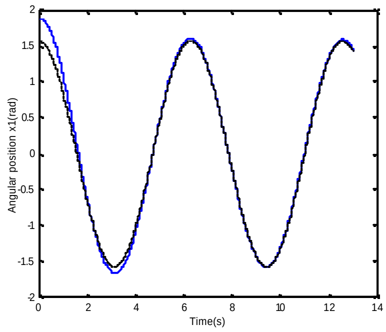

Fig. 17. Angular position, measured (_) and reference(…)(rad)

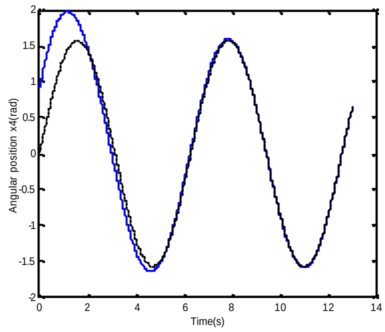

Fig. 18. Angular position, measured (_) and reference(…) (rad)

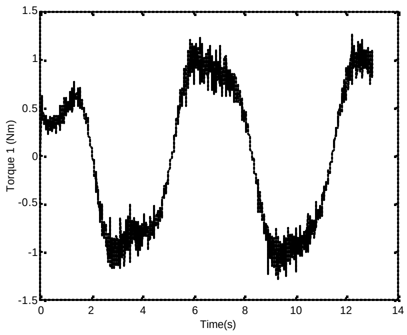

Fig.19. Joint 1 torque

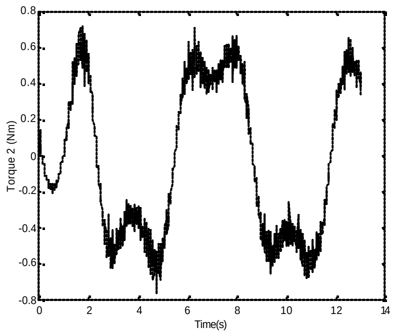

Fig.20. Joint 2 torque

The tracking of position of the tip of the robot was acceptably good. This fact proves the effectiveness of the proposed sliding mode controller based on the selected sliding surface by LMIs.

6 Conclusion

This paper has proposed a technique for controller design using the sliding surface concept and LMIs. As an example, we consider the control of the tip position of a two links robot. The simulations show satisfactory results when the sliding mode controller using the proposed sliding surface is applied.

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