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# Continuous-discrete time observer for a class of state affine nonlinear systems: application to an induction motor

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***Abstract*—The present work proposes a new approach for observer design for a general class of state affine nonlinear systems with discrete-time measurements. A new high gain observer design is developed and analyzed under insightful conditions. This result is achieved by considering a persistent excitation condition that can be validated on-line. The algorithm is applied to an induction motor and validated with simulations.**

## I. INTRODUCTION

A fundamental problem in control design is the need for the state vector of the system to be available in continuous-time. However, this is not always the case, for example when the output measurements are shared through a digital communication network which is available only at discrete-time instants $t_k$. A useful solution is to use a state observer that produces an estimation of the state vector with discrete-time output measurements and whose estimation asymptotically approaches the system state.

Many works have been devoted to the observer design during the last decades when the output is continuously available ([2], [4], [9]). The linear class has already been tackled [19]. For the nonlinear case, many approaches focuses on searching different coordinates where the system can be expressed as a linear one [8], [17], [20], [24], they assume that the observability does not depend on the input. More general classes have been considered such as state affine systems in [10], non uniformly observable systems in [5] where a persistent excitation condition on the input is required.

Several approaches have been proposed for the design of observers for nonlinear systems with discrete-time output measurements. An observer is given in [22] for a class of state affine nonlinear systems with discrete-time output measurements where the notion of universal input is considered. An high gain observer has been presented in [12] for a class of MIMO nonlinear system, this observer works in two times, it mixes : a continuous-time high gain observer and a prediction between sampling times, this allows to dynamically update the observer gain. This work was extended in [21] and an application was presented in [3]. More recently, in [18], an high gain continuous-discrete time observer was proposed using constant observation gains. An approach proposing an appropriate reformulation of the continuous gain but adapted to the sampling constraints has been proposed in [7] for uncertain systems and in [13] for simultaneous state and parameter estimation.

The aim of this paper is to present a continuous-discrete time observer for a general class of MIMO state affine nonlinear systems. This class is more general than the one previously considered. It has been introduced in [6] for continuous time measurements. The main issue is the fact that observability depends on both the input and the state. Thus a new definition of persistent excitation condition has to be introduced. The observer gain is issued from a Lyapunov ODE that is computed in continuous-time. The proposed continuous-discrete time observer is obtained through a redesigned of an high gain continuous-time observer. The observer gain is updated at sampling times when new measurements are available. The convergence analysis is direct and provides insightful expressions of the upper bound on the sampling partition.

The paper is organized as follow. Firstly, the class of considered system is presented together with some assumptions required for the continuous time observer design case. Section III provides an high gain observer for the case of continuously available measurements. The main results, that is, the continuous-discrete time observer is presented in section IV. The proposed observer is applied to a case study, i.e., an induction motor in section V, where simulations are provided. Finally, section VI. concludes this work.

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## II. Problem statement

Let us consider the following class of multi-variable state affine nonlinear system

$$\begin{cases} \dot{x}(t) = A\left(u(t), x(t)\right) x(t) + \varphi\left(x(t), u(t)\right) \\ y(t_k) = Cx\left(t_k\right) = x^1(t_k) \end{cases} \quad (1)$$

with

$$A(u(t), x(t)) = \begin{bmatrix} 0 & A_1(u(t), x^1(t)) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & & A_{q-1}(u(t), x^1(t), \cdots, x^{q-1}(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\varphi(u(t), x(t)) = \begin{pmatrix} \varphi_1(u(t), x^1(t)) \\ \varphi_2(u(t), x^1(t), x^2(t)) \\ \vdots \\ \varphi_{n-1}(u(t), x^1(t), \cdots, x^{q-1}(t)) \\ \varphi_n(u(t), x(t)) \end{pmatrix}$$

$$C = \begin{bmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} & \cdots & 0_{n_1 \times n_q} \end{bmatrix}$$

where the state $x(t) = (x^1 ... x^q)^T \in \mathbb{R}^n$, $x^k \in \mathbb{R}^{n_k}$, $k = 1, ..., q$ with $n_1 = p$ and $\sum_{k=1}^{q} n_k = n$ and each $A_k(u, x)$ is a $n_k \times n_{k+1}$ matrix which is triangular w.r.t. $x$ i.e. $A_k(u, x) = A_k(u, x^1, ..., x^k)$, $k = 1, ..., q-1$; $\varphi(x(t), u(t))$ is a nonlinear vector function that has a triangular structure w.r.t. $x$; $u \in \mathbb{R}^s$ denotes the system input and $y(t_k) \in \mathbb{R}^p$ is the discrete-time output. Furthermore $0 \le t_0 < \cdots < t_k < \ldots$, $\Delta_k = t_{k+1} - t_k$ and $\lim_{k \to +\infty} t_k = +\infty$, we assume that there exists $\Delta_M > 0$ such that $0 < \Delta_k < \Delta_M$, $\forall k \ge 0$.

In order to design an observer for system (1), the following classical assumptions are made (see [5] and [14])

- **A1** The state $x(t)$ and the control $u(t)$ are bounded, i.e., $x(t) \in X$ and $u(t) \in U$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact sets.
- **A2** The functions $A(u(t), x(t))$ and $\varphi\left(x(t), u(t)\right)$ are Lipschitz w.r.t. $x$ uniformly w.r.t. $u$ where $(u, x) \in U \times X$. Their Lipschitz constants are denoted by $L_A$ and $L_\varphi$.

Since the state is confined to the bounded set $X$, one can assume the Lipschitz prolongations of the nonlinearities, using smooth saturation functions (see [23]). In the following, one assumes that the prolongations have been carried out and that the functions $A(u(t), x(t))$ and $\varphi\left(x(t), u(t)\right)$ are provided from these prolongations. This allows to conclude that for any bounded input $u \in U$, the functions $A(u, x)$ and $\varphi\left(x, u\right)$ are globally Lipschitz w.t.r. $x$ and are bounded for all $x \in \mathbb{R}^n$.

Assumption **A1** gives the existence of an upper bound for the state and for $A(x(t), u(t))$, these are defined as

$$x_M = \sup_{t \ge 0} \|x(t)\|. \quad (2)$$

$$\tilde{a} = \sup_{t \ge 0} \|A(u(t), x(t))\| \quad (3)$$

In order to design an high gain observer, one introduces the diagonal matrix $\Delta_\theta$ as

$$\Delta_\theta = diag\begin{bmatrix} I_{n_1} & I_{n_2}/\theta & \cdots & I_{n_q}/\theta^{q-1} \end{bmatrix} \quad (4)$$

The following properties are straightforward, given the structure of $A$ and $C$

$$\Delta_\theta A(u, x) \Delta_\theta^{-1} = \theta A(u, x) \qquad C\Delta_\theta^{-1} = C \quad (5)$$

## III. Continuous-time observer

The observer design for system (1) with a continuous-time output is provided is now considered. System (1) can then be rewritten as

$$\begin{cases} \dot{x}(t) = A\left(u(t), x(t)\right) x(t) + \varphi\left(x(t), u(t)\right), \\ y(t) = Cx\left(t\right) = x^1(t). \end{cases} \quad (6)$$

The candidate observer is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A(u(t), \hat{x}(t))\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\ &\quad -\theta\Delta_\theta^{-1} S^{-1}(t) C^T \left(C\hat{x}(t) - y(t)\right) \end{aligned} \quad (7)$$

where $\hat{x} = (\hat{x}^1 \ldots \hat{x}^q)^T \in \mathbb{R}^n$ with $\hat{x}^k \in \mathbb{R}^{n_k}$, $u$ and $y$ are respectively the input and the output of the system (6) and $S(t)$ is a SPD (Symmetric Positive Definite) matrix governed by the following Lyapunov ODE (Ordinary Differential Equation)

$$\dot{S}(t) = \theta\left(-S(t) - A(u_0(t), \hat{x}_0(t))^T S - SA(u_0(t), \hat{x}_0(t)) + C^T C\right) \quad (8)$$

with $S(0) = S^T(0) > 0$ and $\theta > 0$ is a scalar design parameter.

The main difficulty for designing an observer for system (6) is that it is not necessarily uniformly observable. Indeed, its observability depends on the input and the state, then observability for arbitrarily short times must be ensured. A specific excitation is thus required for the observer design. In order to express this new assumption, we first need to introduce $\Phi(t, s)$, the state transition matrix of the state affine system

$$\dot{\xi}(t) = A(u(t), \hat{x}(t))\xi(t) \quad (9)$$

where $\xi \in \mathbb{R}^n$, $u$ and $\hat{x}$ are respectively the input and the state of the dynamical system (7). The matrix $\Phi_{u,\hat{x}}(t, s)$ is defined as

$$\frac{d\Phi_{u,\hat{x}}(t, s)}{dt} = A(u(t), \hat{x}(t))\Phi_{u,\hat{x}}(t, s), \quad \forall t \ge s \ge 0, \quad (10)$$

$$\Phi_{u,\hat{x}}(t, t) = I_n, \quad \forall t \ge 0, \quad (11)$$

where $I_n$ denotes the identity matrix in $\mathbb{R}^{n \times n}$.

The additional hypothesis can now be expressed

- **A3** The input $u$ is such that for any trajectory $\hat{x}$ of system (7) starting from $\hat{x}(0) \in X$, $\exists \theta^* > 0$, $\exists \delta_0 > 0$, $\forall \theta \ge \theta^*$ and $\forall t \ge 1/\theta$, the following persistent excitation condition is satisfied

$$\int_{t-1/\theta}^{t} \Phi_{u,\hat{x}}(s, t)^T C^T C \Phi_{u,\hat{x}}(s, t) ds \ge \frac{\delta_0}{\theta\alpha(\theta)} \Delta_\theta^2 \quad (12)$$

where $\alpha(\theta) \ge 1$ is a function satisfying

$$\lim_{\theta \to \infty} \frac{\alpha(\theta)}{\theta^2} = 0 \quad (13)$$

***Theorem 1:*** Consider system (6), satisfying assumptions **A1-A2**. Then, for every bounded input satisfying assumption **A3**, there exists a constant $\theta^*$ such that for every $\theta > \theta^*$, system (7) is a state observer for system (6) with an exponential error convergence to the origin for sufficiently high values of $\theta$, i.e. for any initial conditions $(x(0), \hat{x}(0)) \in X$, the observation error $\hat{x}(t) - x(t)$ tends to zero exponentially when $t \to \infty$.

**Proof of Theorem 1** We shall first show that the matrix $S(t)$ is SPD and we shall derive a lower bound for its smallest eigenvalue. Indeed, one can show that the transition matrix, $\tilde{\Phi}_{u,\hat{x}}$ of the following state affine system

$$\dot{\xi}(t) = \theta A(u(t), \hat{x}(t))\xi(t) \tag{14}$$

is given by

$$\tilde{\Phi}_{u,\hat{x}} = \Delta_\theta \Phi_{u,\hat{x}}(t,s)\Delta_\theta^{-1} \tag{15}$$

where $\Phi_{u,\hat{x}}$ is defined by (10).
As a result, the matrix $S(t)$, solution of ODE (8), can be expressed as

$$\begin{aligned} S(t) &= e^{-\theta t}\tilde{\Phi}^T_{u,\hat{x}}(0,t)S(0)\tilde{\Phi}_{u,\hat{x}}(0,t) \\ &\quad + \theta\int_0^t e^{-\theta(t-s)}\tilde{\Phi}^T_{u,\hat{x}}(s,t)C^TC\tilde{\Phi}_{u,\hat{x}}(s,t)ds \\ &= e^{-\theta t}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(0,t)\Delta_\theta S(0)\Delta_\theta\Phi_{u,\hat{x}}(0,t)\Delta_\theta^{-1} \\ &\quad + \theta\int_0^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)\Delta_\theta C^TC\Delta_\theta\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \end{aligned} \tag{16}$$

Using the fact that $C\Delta_\theta = C$ and since $S(0)$ is SPD, one gets for $t \geq 1/\theta$

$$\begin{aligned} S(t) &\geq \theta\int_0^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^TC\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \\ &\geq \theta\int_{t-\frac{1}{\theta}}^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^TC\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \\ &\geq \theta e^{-1}\int_{t-\frac{1}{\theta}}^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^TC\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \end{aligned} \tag{17}$$

$$\geq \theta e^{-1}\frac{\delta_0}{\alpha(\theta)}I_n \tag{18}$$

where $\delta_0$ and $\alpha(\theta)$ are given by assumption **A3**. According to inequality (18), one clearly has

$$\lambda_{\min}(S) \geq \frac{e^{-1}\delta_0}{\alpha(\theta)} \tag{19}$$

We shall now show that $\lambda_{\max}(S)$ is bounded with an upper bound independent of $\theta$. To this end, we shall show that this property is satisfied for each entry of the matrix $S(t)$. Indeed, let us denote by $S_{i,j}$ the block entry of matrix $S$ located at the row $i$ and the column $j$. Then, according to equation (8), one has

$$\dot{S}_{11} = -\theta(S_{11}(t) - I_p) \tag{20}$$

$$\dot{S}_{1j} = -\theta(S_{1j}(t) + S_{1,j-1}(t)A_{j-1}(u(t),\hat{x}(t))) \tag{21}$$

$$j = 2, ..., n$$

$$\begin{aligned} \dot{S}_{ij} &= -\theta(S_{ij}(t) + S_{i,j-1}(t)A_{j-1}(u(t),\hat{x}(t)) \\ &\quad + A^T_{i-1}(u(t),\hat{x}(t))S_{i-1,j}(t)) \\ &\quad i = 2, ..., n, j = i, ..., n \end{aligned} \tag{22}$$

According to (20), one has

$$\begin{aligned} \|S_{11}(t)\| &\leq e^{-\theta t}\|S_{11}(0)\| + \theta\int_0^t e^{-\theta(t-s)}\|I_p\|ds \\ &\leq \|S_{11}(0)\| + \theta\int_0^t e^{-\theta(t-s)}ds \\ &= \|S_{11}(0)\| + 1\left(1 - e^{-\theta t}\right) \leq \|S_{11}(0)\| + 1 \end{aligned} \tag{23}$$

Now, for $j \geq 2$, let us proceed by induction on $j$ in order to show that $S_{1j}$ is bounded with a bound that does not depend on $\theta$. Indeed assume that $S_{1,j-1}$ is bounded and let us denote

$$S_M = \sup_{t\geq 0}\|S_{1,j-1}(t)\| \tag{24}$$

Recall that according to assumptions **A1** and **A2**, the matrices $A_k(u,\hat{x}), k = 0, ..., q-1$ are bounded. Thus, setting

$$A_M = \sup_{t\geq 0}\|A_k(u(t),\hat{x}(t))\| \tag{25}$$

and using (21), one gets for $j = 2, ..., n$

$$\begin{aligned} \|S_{1j}(t)\| &\leq e^{-\theta t}\|S_{1j}(0)\| \\ &\quad + \theta\int_0^t e^{-\theta(t-s)}\|S_{1,j-1}(s)A_{j-1}(u(t),\hat{x}(t))\|ds \\ &\leq \|S_{1j}(0)\| + \theta S_M A_M\int_0^t e^{-\theta(t-s)}ds \\ &= \|S_{1j}(0)\| + S_M A_M\left(1 - e^{-\theta t}\right) \\ &\leq \|S_{1j}(0)\| + S_M A_M \end{aligned} \tag{26}$$

At this step, we have shown that all the entries located at the first row (and the first column) of the symmetric matrix $S$ are bounded. We shall proceed by induction on the row number $i$ in order to show that all the entries of a row are bounded. Indeed, suppose that all the entries of the rows 1 to $i-1$ are bounded (with a bound that does not depend on $\theta$) and let us show that all the entries of the row $i$ are also bounded. Since $S$ is symmetric, all the entries $S_{i,j}$ with $i < j$ are bounded by the induction assumptions. In particular, one has $S_{i-1,i} = \left(S^T\right)_{i,i-1}$ and these matrices are bounded with an upper bound, say $S_M$ independent of $\theta$. Now, we shall show that $S_{i,j}$ is bounded for $j \geq i$. Indeed, according to (22), one has

$$\begin{aligned}\|S_{ii}(t)\| &\le e^{-\theta t}\|S_{ii}(0)\| \\ &\quad + \theta \int_0^t e^{-\theta(t-s)} \|S_{i,i-1}(s)A_{j-1}(u(s),\hat{x}(s)) \\ &\qquad + A_{j-1}^T(u(s),\hat{x}(s))S_{i-1,i}(s)\| ds \\ &\le e^{-\theta t}\|S_{ii}(0)\| + \theta(2S_M A_M)\int_0^t e^{-\theta(t-s)}ds \\ &= \|S_{ii}(0)\| + 2S_M A_M\left(1-e^{-\theta t}\right) \\ &\le \|S_{ii}(0)\| + 2S_M A_M \end{aligned} \tag{27}$$

Now, assume that $S_{i,j-1}$ is bounded and let us show that $S_{i,j}$ is also bounded. Indeed, according to the induction assumptions, the entry $S_{i-1,j}$ located at the row $i-1$ is bounded. From equation (22), one gets

$$\begin{aligned}\|S_{ij}(t)\| &\le e^{-\theta t}\|S_{ij}(0)\| \\ &\quad + \theta \int_0^t e^{-\theta(t-s)} \|S_{i,j-1}(s)A_{j-1}(u(s),\hat{x}(s)) \\ &\qquad + A_{j-1}^T(u(s),\hat{x}(s))S_{i-1,j}(s)\| ds \\ &\le e^{-\theta t}\|S_{ij}(0)\| + \theta(2S_M A_M)\int_0^t e^{-\theta(t-s)}ds \\ &= \|S_{ij}(0)\| + 2S_M A_M\left(1-e^{-\theta t}\right) \\ &\le \|S_{ij}(0)\| + 2S_M A_M \end{aligned} \tag{28}$$

To summarize, we have shown that all the entries of the matrix $S(t)$ are bounded with an upper bound independent of $\theta$. As a result the largest eigenvalues of $S(t)$, $\lambda_{\max}(S)$, is also independent of $\theta$.

Now, we prove the exponential convergence to zero of the observation error. Set $\bar{x} = \Delta_\theta \tilde{x}$ where $\tilde{x} = \hat{x} - x$ is the observation error, the errors dynamics is given by

$$\begin{aligned}\dot{\bar{x}}(t) &= \theta\left[A(u(t),\hat{x}(t)) - S^{-1}(t)C^TC\right]\bar{x}(t) \\ &\quad + \Delta_\theta\left[\tilde{A}(u(t),\hat{x}(t),x(t))x + \tilde{\varphi}(u(t),\hat{x}(t),x(t))\right]\end{aligned} \tag{29}$$

where

$$\tilde{A}(u(t),\hat{x}(t),x(t)) = A(u(t),\hat{x}(t)) - A(u(t),x(t)) \tag{30}$$

$$\tilde{\varphi}(u(t),\hat{x}(t),x(t)) = \varphi(u(t),\hat{x}(t)) - \varphi(u(t),x(t)) \tag{31}$$

Let $V(t) = \bar{x}^T(t)S(t)\bar{x}(t)$ be the Lyapunov candidate function, using (8), one gets

$$\begin{aligned}\dot{V}(\bar{x}(t)) &= -\theta\bar{x}^T(t)S(t)\bar{x}(t) - \theta\bar{x}^T(t)C^TC\bar{x}(t) \\ &\quad + 2\bar{x}(t)S(t)\Delta_\theta\left[\tilde{A}(u(t),\hat{x}(t),x(t))x + \tilde{\varphi}(u(t),\hat{x}(t),x(t))\right]\end{aligned} \tag{32}$$

Proceeding as in [16], one can show that for $\theta > 0$

$$\|2\bar{x}(t)S(t)\Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x(t)\| \le 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\times V(t)L_{\tilde{A}}x_M \tag{33}$$

$$\|2\bar{x}(t)S(t)\Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t))\| \le 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}V(t)L_{\tilde{\varphi}} \tag{34}$$

where $L_{\tilde{A}}$ and $L_{\tilde{\varphi}}$ come from Assumption **A2** and considering $\lambda_{\min}(S)$ as in (19).

By substituting (33) and (34) in (32), one gets

$$\dot{V}(t) \le -\theta\left(1 - 2\sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{n\lambda_{MAX}(S)e}{\delta_0}}\left(L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right)\right)V(t) \tag{35}$$

According to (13), $V(t)$ exponentially converges to zero for values of $\theta$ sufficiently large. This ends the proof.

## IV. CONTINUOUS-DISCRETE TIME OBSERVER

In this section, the main result of the paper is presented, that is, observer design when the output is available at discrete time only. In order to tackle this problem, the observer proposed in section III is redesigned.

The continuous-discrete time candidate observer for system (1) is then defined as follow

$$\dot{\hat{x}}(t) = A(u(t),\hat{x}(t))\hat{x} + \varphi(u(t),\hat{x}(t)) - \theta\Delta_\theta^{-1}S^{-1}(t)C^T\eta(t) \tag{36}$$

$$\dot{S}(t) = \theta\left(-S(t) - A(u(t),\hat{x}(t))^TS - SA(u(t),\hat{x}(t)) + C^TC\right) \tag{37}$$

$$\dot{\eta}(t) = -\theta CS^{-1}(t)C^T\eta(t) \qquad t\in[t_k,t_{k+1}[\,, k\in\mathbb{N} \tag{38}$$

$$\eta(t_k) = C\hat{x}(t_k) - y(t_k) \qquad t = t_k \tag{39}$$

where $\hat{x} = [\hat{x}^1,...,\hat{x}^q]^T$ is the state estimate and $\Delta_\theta$ is the block-diagonal matrix defined in (4) with $\theta > 0$.

We can now present our main result

***Theorem 2:*** Consider system (1), satisfying Assumptions **A1-A2**. Assume that the input $u$ is bounded and fulfill assumption **A3**. Then, there exists $\theta_0 > 0$ such that for every $\theta \ge \theta_0$, there exists $\chi_\theta > 0$ such that if the upper bound of the sampling partition parameter $\Delta_M$ is chosen such that

$$\Delta_M < \chi_\theta \tag{40}$$

then the state of continuous-discrete time observer with discrete-time measurements (36)-(39) exponentially converges to the state of the state affine nonlinear system (1).

**Proof of Theorem 2** Let us now prove the exponential convergence to zero of the observation error. Set $\bar{x} = \Delta_\theta\tilde{x}$ where $\tilde{x} = \hat{x} - x$, the error equation is given by

$$\begin{aligned}\dot{\bar{x}}(t) &= \theta A(u(t),\hat{x}(t))\bar{x}(t) - \theta S^{-1}(t)C^T\eta(t) \\ &\quad + \Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x(t) + \Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t)) \\ &= \theta\left[A(u(t),\hat{x}(t)) - S^{-1}(t)C^TC\right]\bar{x}(t) + \theta S^{-1}(t)C^Tz(t) \\ &\quad + \Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x + \Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t))\end{aligned} \tag{41}$$

where $\tilde{A}(u(t),\hat{x}(t),x(t))$ and $\tilde{\varphi}(u(t),\hat{x}(t),x(t))$ are defined in (30) and (31), respectively, and $z(t) = C\bar{x}(t) - \eta(t)$.

Using the fact that $\eta(t)$ is governed by the ODE (38), one can show that

$$\begin{aligned}\dot{z}(t) &= C[\theta A(u(t),\hat{x}(t))\bar{x}(t) + \Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x(t) \\ &\quad + \Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t))]\end{aligned} \tag{42}$$

A candidate Lyapunov function is given by $V(\bar{x}, t) = \bar{x}^T S(t)\bar{x}$. Using (41) one gets

$$\begin{aligned}\dot{V}(\bar{x}(t), t) &= -\theta\bar{x}^T(t)S(t)\bar{x}(t) - \theta\bar{x}^T(t)C^TC\bar{x}(t) \\ &+ 2\theta\bar{x}(t)C^Tz(t) \\ &+ 2\bar{x}(t)S(t)\Delta_\theta[\tilde{A}(u(t), \hat{x}(t), x(t))x \\ &+ \tilde{\varphi}(u(t), \hat{x}(t), x(t))]\end{aligned} \tag{43}$$

We shall now obtain an over-valuation of $|z(t)|$, according to equation (42), we have

$$|z(t)| \leq \left(\frac{\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}}{\sqrt{\lambda_{\max}(S)}}\right)\int_{t_k}^{t}\sqrt{V(\bar{x}(s), s)}ds \tag{44}$$

where $x_M$ and $\tilde{a}$ are defined by (2) and (3) respectively. It follows that

$$\begin{aligned}\|2\bar{x}(t)C^Tz(t)\| &\leq 2\frac{(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}})}{\lambda_{\max}(S)} \\ &\times \sqrt{V(\bar{x}(t), t)}\int_{t_k}^{t}\sqrt{V(\bar{x}(s), s)}ds\end{aligned} \tag{45}$$

Combing the equations (43), (33), (34) and (45) gives

$$\begin{aligned}\dot{V}(\bar{x}(t), t) &\leq -\theta V(\bar{x}(t), t) \\ &+ 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right]V(\bar{x}(t), t) \\ &+ 2\theta\frac{(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}})}{\lambda_{\max}(S)} \\ &\times \sqrt{V(\bar{x}(t), t)}\int_{t_k}^{t}\sqrt{V(\bar{x}(s), s)}ds\end{aligned} \tag{46}$$

One can rewrite (46) as

$$\begin{aligned}\frac{d\sqrt{V(\bar{x}(t), t)}}{dt} &\leq -\theta\sqrt{V(\bar{x}(t), t)} \\ &+ \left[\frac{2\theta\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)}{\lambda_{\max}(S)}\int_{t_k}^{t}\sqrt{V(\bar{x}(s), s)}ds\right] \\ &+ 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right]\sqrt{V(\bar{x}(t), t)} \\ &\leq -a_\theta\sqrt{V(\bar{x}(t), t)} + b_\theta\int_{t_k}^{t}\sqrt{V(\bar{x}(s), s)}ds\end{aligned} \tag{47}$$

where

$$a_\theta = \theta - 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right] \tag{48}$$

$$b_\theta = \frac{2\theta\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)}{\lambda_{\max}(S)} \tag{49}$$

Applying **Lemma 1** in [15] with $a = a_\theta$ and $b = b_\theta$ and defining $\chi_\theta = a_\theta/b_\theta$ gives the result.

## V. Example

To show the performances of the proposed approach, the model of an induction motor is considered. This type of motor is used in a wide field of industrial applications due to its operating profitability. However, this machine dynamics are highly nonlinear, besides not all its variables are usually available, such as rotor variables. A solution is to use a nonlinear observer to estimate all the states from available measurements. The induction motor can represented by a system of ordinary differential equations, given by (see [11], [1]):

$$\begin{cases}\dot{i}_{s\alpha} = -\gamma i_{s\alpha} + \frac{K}{T_r}\psi_{r\alpha} + pK\omega\psi_{r\beta} + \frac{1}{\sigma L_s}\upsilon_{s\alpha} \\ \dot{i}_{s\beta} = -\gamma i_{s\beta} - pK\omega\psi_{r\alpha} + \frac{K}{T_r}\psi_{r\beta} + \frac{1}{\sigma L_s}\upsilon_{s\beta} \\ \dot{\omega} = \frac{pM}{JL_r}\left(i_{s\beta}\psi_{r\alpha} - i_{s\alpha}\psi_{r\beta}\right) - \frac{C_{res}}{J} \\ \dot{\psi}_{r\alpha} = \frac{M}{T_r}i_{s\alpha} - \frac{1}{T_r}\psi_{r\alpha} - p\omega\psi_{r\beta} \\ \dot{\psi}_{r\beta} = \frac{M}{T_r}i_{s\beta} + p\omega\psi_{r\alpha} - \frac{1}{T_r}\psi_{r\beta}\end{cases} \tag{50}$$

with constant terms

$$T_r = \frac{L_r}{Rr} \quad \sigma = 1 - \frac{M^2}{L_sL_r} \quad K = \frac{M}{\sigma L_sLr} \quad \gamma = \frac{R_s}{\sigma L_s} + \frac{R_rM^2}{\sigma L_sL_r^2}$$

where $i_{s\alpha}$ and $i_{s\beta}$ denote the stator currents; $\psi_{s\alpha}$ and $\psi_{s\beta}$ denote the rotor fluxes and $\omega$ the angular velocity; the input voltages are denoted $\upsilon_{s\alpha}$ and $\upsilon_{s\beta}$. The constant values used in this simulation are presented in table I, [1].

TABLE I
PARAMETERS OF THE MOTOR INDUCTION

| Parameter | Notation | Value |
|---|---|---|
| Poles pairs | $p$ | 2 |
| Stator Resistance | $R_s$ | $9.65\Omega$ |
| Rotor Resistance | $R_r$ | $4.3047\Omega$ |
| Stator Inductance | $L_s$ | $0.4718H$ |
| Rotor Inductance | $L_r$ | $0.4718H$ |
| Mutual Inductance | $M$ | $0.4475H$ |
| Rotor Inertia | $J$ | $0.0293kg/m^2$ |
| Resistance Torque | $C_{res}$ | $0Nm$ |

The nonlinear model (50) is transformed as a state affine nonlinear system of the form (1), as follows:

$$\begin{bmatrix}\dot{x}_1^1 \\ \dot{x}_2^1 \\ \dot{x}_3^1 \\ \dot{x}_2^2 \\ \dot{x}_2^2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & \frac{K}{Tr} & pKx_3^1 \\ 0 & 0 & 0 & -pKx_3^1 & \frac{K}{Tr} \\ 0 & 0 & 0 & \frac{pM}{JL_r}x_2^1 & -\frac{pM}{JL_r}x_1^1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1^1 \\ x_2^1 \\ x_3^1 \\ x_1^2 \\ x_2^2\end{bmatrix} + \begin{bmatrix}-\gamma x_1^1 + \frac{1}{\sigma Ls}\upsilon_{s\alpha} \\ -\gamma x_2^1 + \frac{1}{\sigma Ls}\upsilon_{s\beta} \\ -\frac{C_{res}}{J} \\ \frac{M}{Tr}x_1^1 - \frac{1}{Tr}x_1^2 - px_2^2x_3^1 \\ \frac{M}{Tr}x_2^1 - \frac{1}{Tr}x_2^2 - px_1^2x_3^1\end{bmatrix} \tag{51}$$

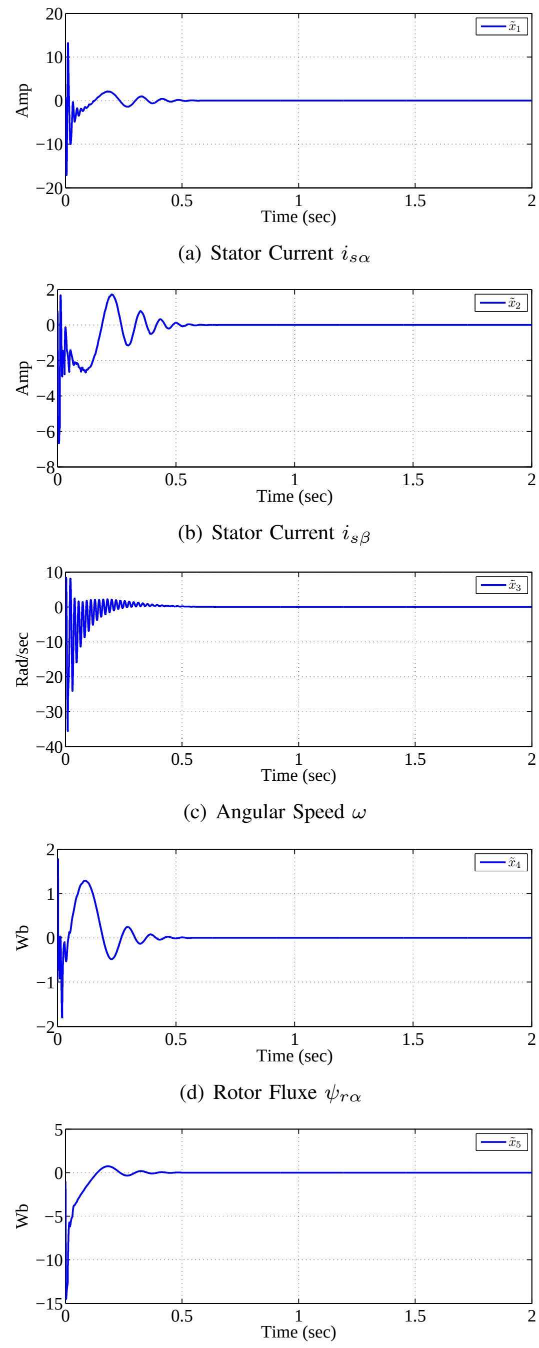
$$y(t_k) = \begin{bmatrix}x_1^1(t_k) & x_2^1(t_k) & x_3^1(t_k)\end{bmatrix}^T \tag{52}$$

where the state is denoted by $x = [x^1 \ x^2]^T$. Thus the variables are gathered as $x^1 = [x_1^1 \ x_2^1 \ x_3^1]^T = [i_{s\alpha} \ i_{s\beta} \ \omega]^T$ and $x^2 = [x_1^2 \ x_2^2]^T = [\psi_{r\alpha} \ \psi_{r\beta}]^T$. We can now design a continuous-discrete time nonlinear observer of the form (36)-(39).

The nonlinear model simulation is carried out with the following values of the inputs: $v_{s\alpha} = 310\cos(wt)$ and $v_{s\beta} = 310\sin(wt)$, where $\omega = 377rad$; the sampled measurement output $y(t_k)$ are available with a constant sampling time $\Delta_k = 10ms$, $k \in \mathbb{N}^*$. The initial conditions are chosen as $x(0) = [x_1^1\ x_2^1\ x_3^1\ x_1^2\ x_2^2] = [0.2\ 0.2\ 10\ 1\ 1]$, $\hat{x}(0) = [\hat{x}_1^1\ \hat{x}_2^1\ \hat{x}_3^1\ \hat{x}_1^2\ \hat{x}_2^2] = [1\ 1\ 0\ 0\ 0]$ and $S(0) = I_{5\times5}$. The simulation results of the continuous-discrete time observer (36)-(39) for the induction motor (51) are presented on Fig. 1. The tuning parameter is taken as $\theta = 10$, then $\Delta_\theta = [I_{3\times3}\quad(1/\theta)I_{2\times2}]$. We can note that the estimations quickly converge to the unknown states $x^2$ thereby confirming the presented results.

## VI. Conclusion

The problem of continuous-discrete time observer has been considered for a general class of non uniformly observable state affine nonlinear systems. A new approach for continuous-discrete time observer design is presented for a discrete time output based on an high gain structure. The proposed design has been applied on simulation for a well known case study of an Induction Motor.




(a) Stator Current $i_{s\alpha}$

(b) Stator Current $i_{s\beta}$

(c) Angular Speed $\omega$

(d) Rotor Fluxe $\psi_{r\alpha}$

(e) Rotor Fluxe $\psi_{r\beta}$

Fig. 1. Estimation error of the Induction Motor state.

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