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Saturated Systems with Sliding Mode Control

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Abstract. *This paper presents a design methodology of robust sliding mode control of systems with the presence of constraint of saturation. The saturation is being on the control matrices. Firstly, we formulate the design of the sliding surface as a problem of root clustering of a reduced system, what leads to the formulation of the new dynamics of the system in sliding mode. Secondly, we realize a selection of a saturated continuous and non-linear control law, to reach the sliding surface. Finally, a numerical application is illustrated to validate the theoretical results of this work.*

Keywords. *Variable Structure Control, Sliding Mode, Saturated Systems, LMI.*

1. Introduction

The saturation phenomenon appears like one of the problems more running in the stability of a system. Nevertheless with the fast evolution of industrial technology, especially in the actuators, it is necessary to envisage methods of resolution for this problem. Used in early days, several methods appeared and are considered to be effective, among them let us quote for examples, the anti-windup design ([8], [11]…), and many other methods which introduce conditions on systems containing saturation functions ([3], [6], [7], [14]…). In robustness terms the sliding mode is a very significant transitory mode for the Variable Structure Control (VSC) ([4], [15], [16], [17]…). Early work was mainly done by Soviet control scientists ([5], [9]…). In recent years, we find more research and many successful applications ([1], [12], [13]…). This paper is organised as follows: in the beginning we present a brief generalities about sliding mode control, after that we present the saturation structure, at end to obtain the new saturated system, with saturation reported on the control vector. In the development which follows, we present the design of the robust saturated control in

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sliding mode. The last stage of this paper is devoted to treat an application of a mechanical system.

2. Sliding Mode Control (SMC) Concepts

The variable structure control is a nonlinear robust control, able to satisfy the requirements of the dynamic characteristics of the system. Indeed the sliding motion is done in a discontinuous way on one or more surfaces of the space of state, called hyper-surfaces. The sliding mode is reached if the system of state crosses and rockroses several time one of these hyper-surfaces. In all the dynamics of the system is forced to be in subspace of the state space. When the sliding mode is reached then $S = Cx = 0$, what explains the insensitivity of this mode for the external and parametric disturbances affecting the dynamics of the system.

The design of the sliding surface is carried out by using the method of the equivalent control, who will allow us in the first time to obtain the reduced system and in the second time to determine the dynamic of this system in sliding mode. Thereafter, the determination of the stabilizing gain allows the calculation of the sliding hyper surface. In particular in this paper the calculation of the gain becomes a simple problem of pole placement.

The design of the control law consists in working out a law of nonlinear control aims at bringing back the trajectory of state on the sliding surface in a finished time t_R and forcing it to remain there. There are many methods for this design, but in this work the scheme adopted is the "unit vector control". The control structure is $u = Lx + \rho \frac{Nx}{\|Mx\| + \delta}$ which offers simplicity of implementation and many advantages, especially in the adjustment of the parameters of synthesis ρ and δ.

3. Problem Statement

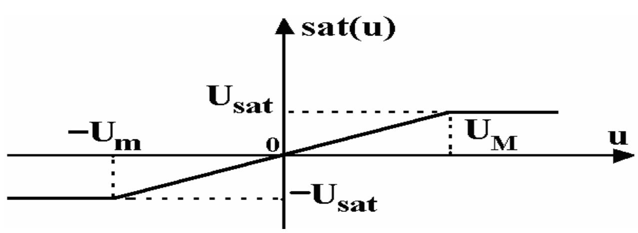

Fig. 1. The structure of the saturation constraint explained well in [10], can be described by the following figure.

Assumption 1: The control vector is subjected to constant limitations in amplitude. It's defined by,

$$u \in \Omega \subset \Re^m = \left\{ u \in \Re^m / -U_m \le u \le U_M ; U_m, U_M > 0 \right\} \tag{1}$$

For $0 < \beta < \frac{1}{2}$ such as $sat(u) = 2\beta u$, the term of saturation sat(u) and β are given by,

$$sat(u) = \begin{cases} \beta(U_M - U_m) & \text{if} \quad u > U_M \\ u & \text{if} \quad -U_m \le u \le U_M \\ \beta(U_m - U_M) & \text{if} \quad u < -U_m \end{cases} \tag{2}$$

with,

$$\beta = \begin{cases} \frac{1}{2}\frac{(U_M - U_m)}{u} & \text{if} \quad u > U_M \\ \frac{1}{2} & \text{if} \quad -U_m \le u \le U_M \\ \frac{1}{2}\frac{(U_m - U_M)}{u} & \text{if} \quad u < -U_m \end{cases} \tag{3}$$

The system can be written as,

$$\dot{x} = Ax + 2\beta Bu \tag{4}$$

with $A \in \Re^{n \times n}, B \in \Re^{n \times m}, x \in \Re^n$ and $u \in \Re^m$.

Assumption 2: The pair (A,B) is controllable, B has full rank m, and $n > m$.

We consider the matrix C, such as the switching surfaces $S = \bigcap_{j=1}^{m} S_j = \left\{ x \in \Re^n : Cx = 0 \right\}$ are usually intersecting hyperplanes passing through the state space origin. The sliding mode occurs when the state reaches and remains in the intersection S of the m hyperplanes. Geometrically, the subspace S is the null space (or Kernel) of C.

Differentiating with respect the time,

$$\dot{S} = C\dot{x} = 0 \tag{5}$$

the substitution of (4) in (5) generates,

$$\dot{S} = CAx + 2C\beta Bu = 0 \tag{6}$$

if $(CB)^{-1}$ exists, then $u_{eq} = -(2\beta CB)^{-1}CAx = -kx$, with,

$$k = (2\beta CB)^{-1}CA \tag{7}$$

The dynamics $\dot{x} = (I - 2\beta B(2\beta CB)^{-1}C)Ax$, describes the motion on the sliding surface which depends only on the choice of the matrix C.

4. Design of the Sliding Surface

In this part we will prove the existence of the sliding mode, indeed the canonical form used by ([4], [15]) for VSC design can be extended to saturated systems to select the gain matrix C.

Assumption 3: There exists an (nxn) orthogonal transformation matrix T such that $y = Tx$ and $TB = [0 \quad B_2]'$ where B has full rank m and B_2 is (mxm) and non-singular.

The transformed state equation is $\dot{y} = TAT^T y + 2\beta TBu$, such as $y^T = [y_1^T \quad y_2^T]$ with $y_1 \in \Re^{n-m}$ and $y_2 \in \Re^m$, can be rewritten as,

$$\begin{aligned} \dot{y}_1 &= A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 &= A_{21}y_1 + A_{22}y_2 + 2\beta B_2 u \end{aligned} \quad \textbf{(8)}$$

Since the sliding condition is $Cx = CT^T y = 0$, with,

$$CT^T = [C_1 \quad C_2] \quad \textbf{(9)}$$

and,

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \textbf{(10)}$$

We can obtain the new defining sliding condition,

$$C_1 y_1 + C_2 y_2 = 0 \quad \textbf{(21)}$$

Assumption 4: CB is non-singular then C_2 must be non-singular.

The sliding mode condition becomes,

$$\begin{aligned} y_2 &= -C_2^{-1}C_1 y_1 = -Fy_1 \\ F &= C_2^{-1}C_1 \end{aligned} \quad \textbf{(32)}$$

with F being an [mx(n-m)] matrix.

The reduced system is (n-m)th order and y_2 becomes a state feedback control. The sliding mode is then governed by,

$$\begin{aligned} \dot{y}_1 &= A_{11}y_1 + A_{12}y_2 \\ y_2 &= -Fy_1 \end{aligned} \quad \textbf{(13)}$$

The closed loop system will then have the dynamics $\dot{y}_1 = [A_{11} - A_{12}F]y_1$. This indicates that the design of a stable sliding mode requires the selection of a matrix F such that $(A_{11} - A_{12}F)$ has (n-m) left-half-plane eigenvalues. Performances can be easily taken into account via root clustering with LMI concept.

If the matrix F has been determined, C is given by,

$$C = [F \quad I_m]T \tag{44}$$

5. Determination of the gain F

To determine the matrix C and the gain F, the method of the LMI seems to us very effective. Indeed to improve the performances of the control law and the response of system. We select to place the poles in a defined area([1], [2]), called area LMI who will allow us to obtain from good result. For that we propose to choose all the eigenvalues of the matrix $(A_{11} - A_{12}F)$ in an area defined by a disc of center q and ray r in the left-half-plane complexes.

To reduce the writing we pose $(A_{11} - A_{12}F) = \Sigma_1$.
(13) is stable and its eigenvalues are localised in the disc, if the two following inequalities are checked,

$$(\Sigma_1 + qI)^T P(\Sigma_1 + qI) - r^2 P < 0 \tag{55}$$

and,

$$P = P^T > 0 \tag{66}$$

The complement of Schur is given by,
Lemma 1: [2] either $Q(x) = Q(x)^T$, $R(x) = R(x)^T$ and $S(x)$ are matrices of dimension $n \times n$ and we suppose that P and R are invertible.

Then the following LMI are equivalent,

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0 \tag{77}$$

with,

$$R(x) > 0, Q(x) - S(x)R(x)^{-1}S(x) > 0 \tag{88}$$

We pose $S = P^{-1}$ and while multiplying pre and post by $\begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix}$, we obtain,

$$\begin{pmatrix} -r^2S & S\Sigma_1^T + qS \\ \Sigma_1 S + qS & -S \end{pmatrix} < 0 \qquad \textbf{(99)}$$

while posing $FS = R$, (19) can be written in the following form,

$$\begin{pmatrix} -r^2S & SA_{11}^T - R^T A_{12}^T + qS \\ A_{11}S - A_{12}R + qS & -S \end{pmatrix} < 0 \qquad \textbf{(20)}$$

This carries out to write the following result,

Theorem 1: The system is stable and its eigenvalues are localised in the disc of center q and ray r, if there is a constant $\alpha > 0$, a matrix R and a matrix S positive and symmetrical, let us know that,

$$\begin{pmatrix} -r^2S & SA_{11}^T - R^T A_{12}^T + qS & 0 \\ A_{11}S - A_{12}R + qS & -S & 0 \\ 0 & 0 & -\alpha I \end{pmatrix} < 0 \qquad \textbf{(21)}$$

Then the gain is given by $F = RS^{-1}$.

6. Saturated Control Law Design

To reach the sliding surface and ensures that trajectories are directed towards the switching surface from any point in the state space, we realise a selection of a saturated feedback non linear control function $u = u_L + u_N$, where u_L and u_N are the linear and non linear control law parts. The general form is the following,

$$u = Lx + \rho \frac{Nx}{\|Mx\| + \delta} \qquad \textbf{(22)}$$

where L, N and M are appropriate matrix, ρ is a design parameter and δ is a smoothing parameter.

We form a second transformation non-singular $T_2 : \Re^n \to \Re^n$ such that $z = \lfloor z_1^T \quad z_2^T \rfloor$ with $z_1^T \in \Re^{n-m}$ and $z_2^T \in \Re^m$,

$$z = T_2 y = T_2 T x \qquad \textbf{(23)}$$

where,

$$T_2 = \begin{bmatrix} I_{n-m} & 0 \\ F & I_m \end{bmatrix} \qquad \textbf{(24)}$$

and,

$$T_2^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -F & I_m \end{bmatrix} \quad \textbf{(25)}$$

with $z_1 = y_1$ and $z_2 = Fy_1 + y_2$. The system equation becomes,

$$\begin{aligned} \dot{z}_1 &= \Sigma_1 z_1 + \Sigma_2 z_2 \\ \dot{z}_2 &= \Sigma_3 z_1 + \Sigma_4 z_2 + 2\,\beta\, B_2 u \end{aligned} \quad \textbf{(26)}$$

with,

$$\begin{aligned} &\Sigma_1 = A_{11} - A_{12}F, \Sigma_2 = A_{12} \\ &\Sigma_3 = F\Sigma_1 + A_{21} - A_{22}F, \Sigma_4 = FA_{12} + A_{22} \end{aligned} \quad \textbf{(27)}$$

Forcing $z_2 = \dot{z}_2 = 0$, the linear control law is given by,

$$u_L = -(2\,\beta\, B_2)^{-1}(\Sigma_3 z_1 + (\Sigma_4 - \Sigma_4^*)z_2) \quad \textbf{(28)}$$

$$u_L = -(2\,\beta\, B_2)^{-1}[\Sigma_3 \quad (\Sigma_4 - \Sigma_4^*)]T_2 Tx \quad \textbf{(29)}$$

$$L = -(2\,\beta\, B_2)^{-1}[\Sigma_3 \quad (\Sigma_4 - \Sigma_4^*)]T_2 T \quad \textbf{(30)}$$

$\Sigma_4^* = \text{diag}\{\mu_i\} \in \Re^{m\times m}$ is a design matrix and $\text{Re}(\mu_i) < 0$ for i=1 to m.
Let the Lyapunov equation be given by,

$$V_2(z_2) = \frac{1}{2} z_2^T P_2 z_2 \quad \textbf{(31)}$$

where P_2 is the unique solution of the Lyapunov equation,

$$P_2\Sigma_4^* + \Sigma_4^* P_2 + I_m = 0 \quad \textbf{(32)}$$

Then $P_2 z_2 = 0$ if and only if $z_2 = 0$,
and we may take,

$$u_N = -\rho \frac{B_2^{-1} P_2 z_2}{\|P_2 z_2\| + \delta} \quad \textbf{(33)}$$

Differentiating Lyapunov equation,

$$\dot{V}_2(z_2) = \frac{1}{2}\dot{z}_2^T P_2 z_2 + \frac{1}{2} z_2^T P_2 \dot{z}_2 \quad \textbf{(34)}$$

After replacing $\dot{z}_2$ and some intermediate calculations we obtain,

$$\dot{V}_2(z_2) = \frac{1}{2}\left[\Sigma_4^* z_2 + 2\beta B_2 u_N^c\right]^T P_2 z_2 + \frac{1}{2} z_2^T P_2 \left[\Sigma_4^* z_2 + 2\beta B_2 u_N^c\right] \quad (35)$$

$$\dot{V}_2(z_2) = z_2^T P_2 2\beta B_2 u_N^c + z_2^T P_2 \Sigma_4^* z_2 \quad (36)$$

While using $P_2\Sigma_4^* = -\left(\frac{I_m}{2}\right)$, we obtain,

$$\dot{V}_2(z_2) = z_2^T P_2 2\beta B_2 u_N^c - z_2^T \left(\frac{I_m}{2}\right) z_2 \quad (37)$$

$$\dot{V}_2(z_2) = -\frac{1}{2}\|z_2\|^2 + z_2^T P_2 2\beta B_2 u_N^c \quad (38)$$

However we have,

$$z_2^T P_2 2\beta B_2 u_N^c = -\rho \frac{\|P_2 z_2\|^2}{\|P_2 z_2\| + \delta} \quad (39)$$

From where $\dot{V}_2(z_2) < 0$, since the existence of the nonlinear component is checked then we can deduce the matrices M and N in original x-space,

$$N = -(2\beta B_2)^{-1}[0 \quad P_2]T_2 T \quad (40)$$

and,

$$M = [0 \quad P_2]T_2 T \quad (41)$$

7. Numerical Application

Let us consider the reduced model of a train, consistent in an engine of mass M_1 and a wagon of mass M_2, mutually connected by a spring of stiffness k. the train moves on flat ground and is slowed down by friction (dynamic coefficient of friction μ). The state variables are the positions (x_1, x_2) and the speeds $(v_1 = \dot{x}_1, v_2 = \dot{x}_2)$. The entry is the control u, introduced as being a controlled force generated by the engine.

Modeling provides the following equation of state,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M_1} & -\mu g & \frac{k}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M_2} & 0 & -\frac{k}{M_2} & -\mu g \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} u \quad (42)$$

We apply the following values to the parameters of system, $M_1 = 10kg$, $M_2 = 0.5kg$, $k = 1N/m$, $\mu = 0.002s/m$ and $g = 9.8m/s^2$.
The entry u is limited by the following framing,

$$-40 \le u \le 20 \quad (43)$$

The initial condition is given by,

$$x_0 = [0.6 \quad 0 \quad 0.3 \quad 0]' \quad (44)$$

After simulation we obtain the following results,

$$F = [5.2472 \quad 1.5876 \quad -4.5148] \quad (45)$$

The matrices of L, M and N are given by,

$$C = [-5.4732 \quad -1 \quad 1.9978 \quad -4.4253] \quad (46)$$

$$L = [-142.2384 \quad -64.5362 \quad 107.4842 \quad -23.4076] \quad (47)$$

$$M = [-2.7366 \quad -0.5 \quad 0.9989 \quad -2.2127] \quad (48)$$

$$N = [-27.3661 \quad -5 \quad 9.9890 \quad -22.1265] \quad (49)$$

Simulations enable us to obtain the various following figures

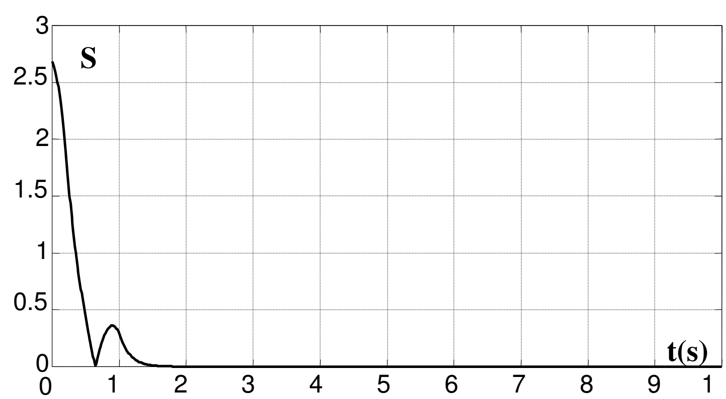

Fig. 4. This figure shows us that the saturated control is able to reach the surface S in a small time, also the surface norms remains small.

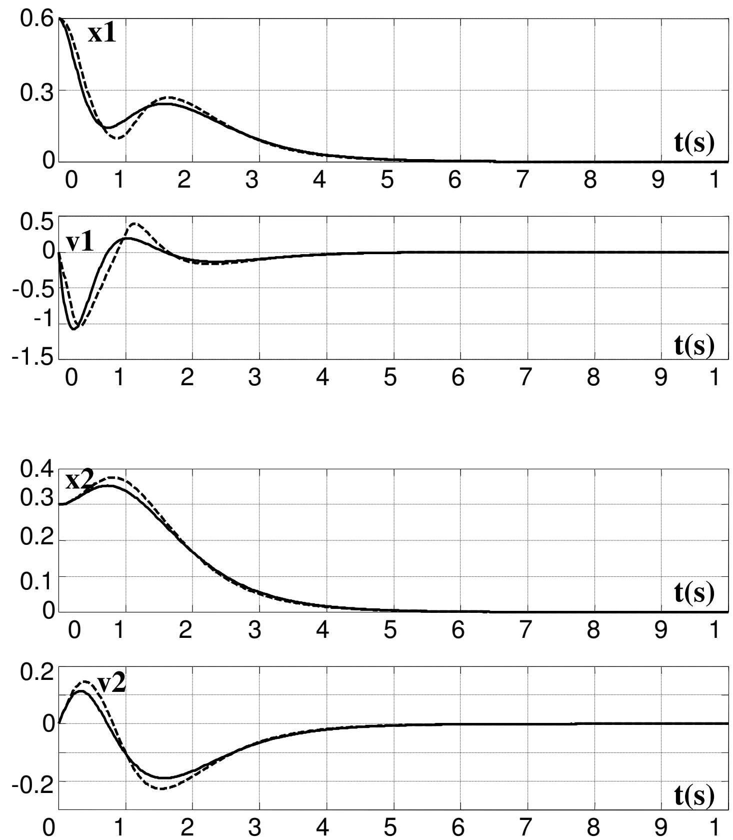

Fig. 1. This figure present, the evolution of state variables (-.-: system with saturation, -: system without saturation constraint). The state variables dynamics of the saturated system have a more slowly transient mode than that of the system without saturation.

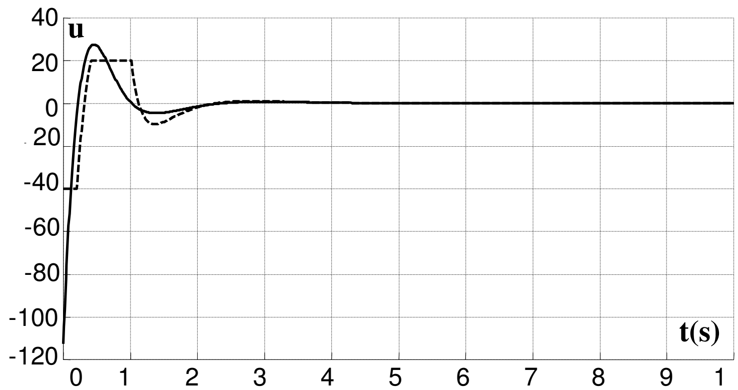

Fig. 3. This figure show the evolution of control input (-.- : system with saturation, - : system without saturation constraint). We can notice that the control is saturated and always inferior to its maximal values. Also it shows a typical stable sliding mode convergence.

8. Conclusion

We presented in this paper a design approach for linear-time invariant saturated system in sliding mode. The control input is saturated and is being of constant limitations in amplitude. In the first step we presented in a general way the sliding mode control concepts, also a brief presentation of saturation structure has been proposed. The design of the variable structure control methodology is exposed then the design of the sliding surface is formulated as a pole assignment of a reduced system. In the second step, we introduce a non-linear saturated control scheme able to reach the sliding surface in a small time t_R and eliminates the undesirable chattering phenomenon what ensures a stable sliding mode motion. Numerical simulation is presented to show the applicability, the efficiency, and the robustness of the proposed method.

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