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# Stability and stabilization of 2D continuous time varying delay systems

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**Abstract.** *This paper deals with the problem of delay-dependent stability and stabilization of $2D$ continuous time varying delay systems described by the Roesser model. Some sufficient conditions ensuring asymptotic stability and stabilization are established in forms of linear matrix inequality (LMI) technique* via *Lyapunov techniques with additional free weighting matrices. A numerical example is introduced to show the efficiency of the proposed criteria for a $2D$ linear time-varying delay system.*
**keywords:** *$2D$ Roesser model, time-varying delay systems, Lyapunov-Krasovskii functional, LMI.*

## 1 Introduction

Delayed multidimensional systems have been recently introduced but in the majority of the existing studies only the discrete case have been analyzed (see e.g. [5, 10, 11, 15, 16]) except for a few recent papers [1, 7, 9] where a Lyapunov approach is applied to continuous Roesser models. These papers consider a constant time delays. Recently, the delay-dependent stability problem for two dimensional systems with time-varying delays has been adressed [4, 13], in the discrete case. However, to the author's knowledge, in the continuous case, this problem has not been fully investigated except a recent paper we have published in [6] where a delay dependent stability criterion is derived for 2D continuous time varying delay systems. It is inspired from ([14, 17]) where some delay-dependent stability criteria, for one dimensional continuous systems, are devised by taking into account the relationship between the terms in the Leibniz-Newton formula by means of a set of free weighting matrices leading to linear matrix inequalities (LMI) conditions.
This paper addresses the problem of stability and stabilization for 2D continuous time varying delay systems. The paper is organized as follows: in section 2, we introduce the mathematical background needed to address the problem. In section 3, we introduce our main results: first, a sufficient condition is derived to check the asymptotic stability of the system using Lyapunov techniques. This delay dependent condition is different from the one presented in [6] and will be shown to be less conservative. In the derivative of the Lyapunov functional, the term

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$\dot{x}(t_1,t_2)$ is retained but the relationship among the term in the system equation is expressed by some free weighting matrices. In consequence, the Lyapunov matrices in the Lyapunov functional are not involved in any product terms with the system matrices. This idea developed in [8] provides some extra freedom in the selection of the weighting matrices, which have the potential to yield less conservative results. Second, we give a delay dependent criterion to design a state feedback controller for 2D continuous time varying delay systems which stabilizes the system. These conditions are expressed in terms of LMIs (linear matrix inequalities, see [3]). Finally section 4 presents an illustrative example to show the effectiveness of the proposed criteria.

**Notations:**

Throughout the paper we will use the following notations: a matrix added to its symmetric will be called $\mathrm{sym}\{A\} = A^T + A$ and $(*)$ in a symmetric matrix denotes the corresponding symmetric element. Also, $0_{n\times m}$ is the $n \times m$ zero matrix, and $I_n$ is the $n \times n$ identity matrix. Some formula will be used in the paper, in particular the Leibniz-Newton formula which is given by

$$x^h(t_1,t_2) - x^h(t_1-\tau_1(t_1),t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \dot{x}^h(s,t_2)ds = 0 \qquad (1)$$

## 2 Problem formulation

The class of 2-D systems with delays under consideration is represented by an extension of the Roesser model (see [12] and [2]) of the form:

$$\begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix} \begin{bmatrix} x^h(t_1-\tau_1(t_1),t_2) \\ x^v(t_1,t_2-\tau_2(t_2)) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1,t_2) \qquad (2)$$

where $x^h(t_1,t_2)$ is the horizontal state in $\mathbb{R}^{n_h}$, $x^v(t_1,t_2)$ is the vertical state in $\mathbb{R}^{n_v}$, $u(t_1,t_2)$ is the control vector in $\mathbb{R}^m$, $\tau_1$ and $\tau_2$ are the delays in horizontal and vertical directions respectively and $A_{ij}$, $A_{ijd}$ and $B_i$, $(i,j=1,2)$, are real constant matrices of appropriate dimensions. The initial conditions are given by

$$\begin{aligned} x^h(\theta,t_2) &= f(\theta,t_2), \forall t_2 \quad \text{and} \quad -h_1 < \theta < 0 \\ x^v(t_1,\theta) &= g(t_1,\theta), \forall t_1 \quad \text{and} \quad -h_2 < \theta < 0 \end{aligned}$$

The time-delays $\tau_1(t_1)$ and $\tau_2(t_2)$ are time-varying continuous functions that satisfy

$$\begin{aligned} 0 < \tau_1(t_1) \leq h_1, \dot{\tau_1}(t_1) \leq d_1 \\ 0 < \tau_2(t_2) \leq h_2, \dot{\tau_2}(t_2) \leq d_2 \end{aligned}$$

where $f$ and $g$ are continuous functions. For such a system we denote

$$x(t_1,t_2) \equiv \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix}, \dot{x}(t_1,t_2) \equiv \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix}$$

$$x(t_1-\tau_1(t_1), t_2-\tau_2(t_2)) \equiv \begin{bmatrix} x^h(t_1-\tau_1(t_1),t_2) \\ x^v(t_1,t_2-\tau_2(t_2)) \end{bmatrix}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, A_d = \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

which allows us to write (2) in the usual form

$$\dot{x}(t_1,t_2) = Ax(t_1,t_2) + A_d x(t_1-\tau_1(t_1), t_2-\tau_2(t_2)) + Bu(t_1,t_2) \tag{3}$$

Consider the state feedback control:

$$u(t_1,t_2) = Kx(t_1,t_2) \tag{4}$$

where the matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

is the state feedback gain to be determined.

## 3 Main results

### 3.1 Asymptotic stability

In this section, we investigate stability condition for time-varying delay system (2), with $u(t_1,t_2) = 0$.

**Theorem 1** *Given matrices* $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} > 0$, $U = H^{-1}$ *and* $W = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} < I$, *the system (2) is asymptotically stable if there exist symmetric positive-definite matrices* $P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0$, $Q = Q^T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} > 0$ *and* $R = R^T = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} > 0$ *and any appropriately dimensioned matrices* $\Upsilon_0$, $\Upsilon_1$, $\Upsilon_2$, $\Lambda_1$, $\Lambda_2$ *and* $\Lambda_3$ *such that the following LMI is verified:*

$$\Phi = \begin{bmatrix} Q - \mathrm{sym}\{\Lambda_1 A\} + \mathrm{sym}\{\Upsilon_0\} & A^T\Lambda_2^T - \Lambda_1 A_d + \Upsilon_{10} & P + \Lambda_1 - A^T\Lambda_3^T + \Upsilon_2^T & HU\Upsilon_0 \\ * & -(I_n - W)Q - \mathrm{sym}\{\Lambda_2 A_d\} - \mathrm{sym}\{\Upsilon_1\} & \Lambda_2 - A_d^T\Lambda_3^T - \Upsilon_2^T & HU\Upsilon_1 \\ * & * & HR + \mathrm{sym}\{\Lambda_3\} & HU\Upsilon_2 \\ * & * & * & -HUR \end{bmatrix} < 0 \tag{5}$$

*with*

$$\Upsilon_0 = \begin{bmatrix} S_0 & 0 \\ 0 & T_0 \end{bmatrix}; \Upsilon_1 = \begin{bmatrix} S_1 & 0 \\ 0 & T_1 \end{bmatrix}; \Upsilon_2 = \begin{bmatrix} S_2 & 0 \\ 0 & T_2 \end{bmatrix}; \Upsilon_{10} = \begin{bmatrix} S_1 - S_0^T & 0 \\ 0 & T_1 - T_0^T \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix}; \Lambda_2 = \begin{bmatrix} N_2 & 0 \\ 0 & M_2 \end{bmatrix}; \Lambda_3 = \begin{bmatrix} N_3 & 0 \\ 0 & M_3 \end{bmatrix}$$

$$X_{11} = \begin{bmatrix} X_{11}^h & 0 \\ * & X_{11}^v \end{bmatrix}; X_{12} = \begin{bmatrix} X_{12}^h & 0 \\ * & X_{12}^v \end{bmatrix}; X_{22} = \begin{bmatrix} X_{22}^h & 0 \\ * & X_{22}^v \end{bmatrix}$$
$$X_{13} = \begin{bmatrix} X_{13}^h & 0 \\ * & X_{13}^v \end{bmatrix}; X_{23} = \begin{bmatrix} X_{23}^h & 0 \\ * & X_{23}^v \end{bmatrix}; X_{33} = \begin{bmatrix} X_{33}^h & 0 \\ * & X_{33}^v \end{bmatrix}$$

*Proof.* Proof of theorem 1 is given in the appendix 6.1.

## 3.2 Stabilization

The objective of this section is the design of a stabilizing state-feedback controller for system (2). Using the state-feedback control (4), (2) can be rewritten as:

$$\dot{x}(t_1, t_2) = A_c x(t_1, t_2) + A_d x(t_1 - \tau_1(t_1), t_2 - \tau_2(t_2)) + Bu(t_1, t_2) \tag{6}$$

where:

$$A_c = \begin{bmatrix} (A_{11} + B_1 K_1) & (A_{12} + B_1 K_2) \\ (A_{21} + B_2 K_1) & (A_{22} + B_2 K_2) \end{bmatrix}$$
$$A_d = \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix}$$

The problem is then to compute a static feedback control given by (4) such that the closed-loop 2D system (6) is asymptotically stable.

**Theorem 2** *Let* $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} > 0$, $U = H^{-1}$ *and* $W = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} < I$ *be given matrices, then the system (2) is stabilizable with the control law (4) if there exist symmetric positive-definite matrices* $X = X^T = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0$, $\bar{P} = \bar{P}^T = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} > 0$, $\bar{Q} = \bar{Q}^T = \begin{bmatrix} \bar{Q}_1 & 0 \\ 0 & \bar{Q}_2 \end{bmatrix} > 0$ *and* $\bar{R} = \bar{R}^T = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix} > 0$ *and any appropriately dimensioned matrices* $\bar{\Upsilon}_0$, $\bar{\Upsilon}_1$, $\bar{\Upsilon}_2$ *and* $Y > 0$ *such that the following LMI is verified:*

$$\begin{bmatrix} \bar{Q} + \mathrm{sym}\{\bar{\Upsilon}_0\} - \mathrm{sym}\{AX\} - \mathrm{sym}\{BY\} & -A_d X - XA^T - Y^T B^T + \bar{\Upsilon}_{10}^T & \bar{\Upsilon}_2^T + X + \bar{P} - XA^T - Y^T B^T & HU\bar{\Upsilon}_0 \\ * & -(I_n - W)\bar{Q} - \mathrm{sym}\{\bar{\Upsilon}_1\} - \mathrm{sym}\{A_d X\} & X - XA_d^T - \bar{\Upsilon}_2^T & HU\bar{\Upsilon}_1 \\ * & * & H\bar{R} + \mathrm{sym}\{X\} & HU\bar{\Upsilon}_2 \\ * & * & * & -HU\bar{R} \end{bmatrix} < 0 \tag{7}$$

*with*

$$\bar{\Upsilon}_0 = \begin{bmatrix} \bar{S}_0 & 0 \\ 0 & \bar{T}_0 \end{bmatrix}; \bar{\Upsilon}_1 = \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{T}_1 \end{bmatrix}; \bar{\Upsilon}_2 = \begin{bmatrix} \bar{S}_2 & 0 \\ 0 & \bar{T}_2 \end{bmatrix}; \bar{\Upsilon}_{10} = \begin{bmatrix} \bar{S}_1 - \bar{S}_0^T & 0 \\ 0 & \bar{T}_1 - \bar{T}_0^T \end{bmatrix}$$

*and* $X = \Lambda^{-1}$, $Y = KX$, $Q = \Lambda\bar{Q}\Lambda$, $R = \Lambda\bar{R}\Lambda$, $P = \Lambda\bar{P}\Lambda$, $\Upsilon_i = \Lambda\bar{\Upsilon}_i\Lambda$, $i = 0, 1, 2$. *The gains* $K_1$ *and* $K_2$ *of the control law (4) are given by*

$$K_1 = Y_1 X_1^{-1}, K_2 = Y_2 X_2^{-1} \tag{8}$$

*Proof.* Proof of Theorem 2 is given in the appendix 6.2.

## 4 Example

In order to show the applicability of our results, consider a 2D continuous system represented by (2) with:

$$A_{11} = \begin{bmatrix} -1.8887 & -1.4069 \\ -0.1447 & -2.1601 \end{bmatrix}, A_{22} = \begin{bmatrix} 2.2169 & -1.0753 \\ 6.0811 & 0.9372 \end{bmatrix},$$
$$A_{12} = \begin{bmatrix} 15.8162 & -6.7649 \\ 4.2121 & 5.0797 \end{bmatrix}, A_{21} = \begin{bmatrix} -0.7902 & -0.0011 \\ -0.4672 & -1.7982 \end{bmatrix}$$

The delay matrices are given by:

$$A_{11d} = \begin{bmatrix} -0.1 & 0 \\ -0.1 - 0.1 & \end{bmatrix}, A_{22d} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}$$
$$A_{12d} = \begin{bmatrix} 0.4 & 0.4 \\ -0.08 & 0.04 \end{bmatrix}, A_{21d} = \begin{bmatrix} -0.24 & 0 \\ 0 & 0.04 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0.3 \\ 0 & 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.3 \end{bmatrix}$$

The parameters $h_1$, $h_2$, $d_1$ and $d_2$ are modified in an iterative process until the LMI (7) was found feasible. The obtained feedback gains

$$K_1 = \begin{bmatrix} -0.1185 & 0.0132 \\ 1.8912 & 0.5704 \end{bmatrix}, K_2 = \begin{bmatrix} -40.0278 & 10.7805 \\ -46.1860 & -29.8926 \end{bmatrix}$$

are then injected to construct the closed loop system which is again checked by condition (5) of Theorem 1. The maximum bounds of delays obtained are

$$h_{1max} = 8.37, \quad h_2 = 3.33, \quad \text{and} \quad d_1 = d_2 = 0.8$$

The condition obtained in [6] applied to the present closed loop system yields the delay bounds $h_{1max} = 0.3273$ and $h_2 = 0.3026$ which illustrate that the delay-dependent condition given in this paper is less conservative than the existing result proposed in [6].

*Remark 1.* From a numerical point of view, it is worth noting that matrices $A_{12}$, $A_{21}$ $A_{12d}$ and $A_{21d}$ could yield badly conditioned LMI's. In addition, the fact that all the matrices are block diagonal increases the possibility that the LMI will be badly conditioned resulting in non feasible LMI.

## 5 Conclusion

To conclude, let us highlight the general contribution of this paper. We first developed a sufficient condition of asymptotic stability for 2D continuous time varying delay systems. Using Lyapunov approach and the Leibniz-Newton formula, we proposed the synthesis of a state feedback controller. The interesting fact in these conditions is that they are delay dependent and expressed in terms of LMIs, so they are tractable from a computational point of view. Finally, a numerical example is provided to illustrate the results.

# 6 Appendix

## 6.1 Proof of Theorem 1

**Lemma 1 ([14]).** *For any semi-positive definite matrix* $X^h = \begin{bmatrix} X_{11}^h & 0 & X_{12}^h & 0 & X_{13}^h & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & X_{22}^h & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & X_{33}^h & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \geq 0$, *the following holds*

$$h_1 \xi^T(t_1,t_2) X^h \xi(t_1,t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \xi^T(t_1,t_2) X^h \xi(t_1,t_2) \geq 0 \tag{9}$$

*where*

$$\xi(t_1,t_2) = [x^{hT}(t_1,t_2) \quad x^{vT}(t_1,t_2) \quad x^{hT}(t_1-\tau_1(t_1),t_2) \quad x^{vT}(t_1,t_2-\tau_2(t_2)) \quad \dot{x}^{hT}(t_1,t_2) \quad \dot{x}^{vT}(t_1,t_2)]^T$$

Let us define

$$V(x(t_1,t_2)) = V_1(t_1,t_2) + V_2(t_1,t_2) \tag{10}$$

as a possible Lyapunov Krasovskii functional candidate for the system (2) with:

$$V_1(t_1,t_2) = x^{hT}(t_1,t_2) P_1 x^h(t_1,t_2) + \int_{t_1-\tau_1(t_1)}^{t_1} x^{hT}(\theta,t_2) Q_1 x^h(\theta,t_2) d\theta + \int_{-h_1}^{0} \int_{t_1+\theta}^{t_1} \dot{x}^{hT}(s,t_2) R_1 \dot{x}^h(s,t_2) ds d\theta$$

$$V_2(t_1,t_2) = x^{vT}(t_1,t_2) P_2 x^v(t_1,t_2) + \int_{t_2-\tau_2(t_2)}^{t_2} x^{vT}(t_1,\theta) Q_2 x^v(t_1,\theta) d\theta + \int_{-h_2}^{0} \int_{t_2+\theta}^{t_2} \dot{x}^{vT}(t_1,k) R_2 \dot{x}^v(t_1,k) dk d\theta$$

The derivative of function $V(x(t_1,t_2))$ along the vector

$$\varsigma(t_1,t_2) = \begin{bmatrix} \dfrac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix}$$

is given by:

$$\begin{aligned} \nabla_\varsigma V(x(t_1,t_2)) = (\nabla V)^T \varsigma(t_1,t_2) &= \begin{bmatrix} \dfrac{\partial V}{\partial x^h} & \dfrac{\partial V}{\partial x^v} \end{bmatrix} \varsigma(t_1,t_2) \\ &= \frac{\partial V(t_1,t_2)}{\partial x^h(t_1,t_2)} \frac{\partial x^h(t_1,t_2)}{\partial t_1} + \frac{\partial V(t_1,t_2)}{\partial x^v(t_1,t_2)} \frac{\partial x^v(t_1,t_2)}{\partial t_2} \\ &= \frac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)} \frac{\partial x^h(t_1,t_2)}{\partial t_1} + \frac{\partial V_2(t_1,t_2)}{\partial x^v(t_1,t_2)} \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{aligned}$$

where $\nabla V$ is the gradient of the function $V$. Let

$$\xi(t_1,t_2) = [x^{hT}(t_1,t_2) \quad x^{vT}(t_1,t_2) \quad x^{hT}(t_1-\tau_1(t_1),t_2) \quad x^{vT}(t_1,t_2-\tau_2(t_2)) \quad \dot{x}^{hT}(t_1,t_2) \quad \dot{x}^{vT}(t_1,t_2)]^T,$$

$$\zeta(t_1,t_2,s,k) = [x^{hT}(t_1,t_2) \quad x^{vT}(t_1,t_2) \quad x^{hT}(t_1-\tau_1(t_1),t_2) \quad x^{vT}(t_1,t_2-\tau_2(t_2)) \quad \dot{x}^{hT}(t_1,t_2) \quad \dot{x}^{vT}(t_1,t_2) \quad \dot{x}^{hT}(s,t_2) \quad \dot{x}^{vT}(t_1,k)]^T$$

and

$$e_i = \left[0_{n\times(i-1)n} \ I_n \ 0_{n\times(8-i)n}\right]^T, i = 1,2,...,8$$

Using the Leibniz-Newton formula (1), we can write

$$x^h(t_1-\tau_1(t_1),t_2) = x^h(t_1,t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \dot{x}^h(s,t_2)ds \tag{11}$$

Then, for any appropriately dimensioned matrices $S_0$, $S_1$ and $S_2$, we have

$$2\left[x^{hT}(t_1,t_2)S_0 + x^{hT}(t_1-\tau_1(t_1),t_2)S_1 + \dot{x}^{hT}(t_1,t_2)S_2\right] \times \left\{x^h(t_1,t_2) - x^h(t_1-\tau_1(t_1),t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \dot{x}^h(s,t_2)ds\right\} = 0 \tag{12}$$

Similarly, for any matrices $N_1$, $N_2$ and $N_3$ of appropriate dimensions, we have

$$2\left[x^{hT}(t_1,t_2)N_1 + x^{hT}(t_1-\tau_1(t_1),t_2)N_2 + \dot{x}^{hT}(t_1,t_2)N_3\right] \times \left\{\dot{x}^h(t_1,t_2) - A_{11}x^h(t_1,t_2) - A_{12}x^v(t_1,t_2) - A_{11d}x^h(t_1-\tau_1(t_1),t_2) - A_{12d}x^v(t_1,t_2-\tau_2(t_2))\right\} = 0. \tag{13}$$

The free weighting matrices $S_i$ $(i = 0,1,2)$ in (12) are used to express the relationship between $x^h(t_1,t_2)$, $x^h(t_1-\tau_1(t_1),t_2)$ and $\int_{t_1-\tau_1(t_1)}^{t_1} \dot{x}^h(s,t_2)ds$, using the Leibniz-Newton formula.

The free weighting matrices $N_i$ $(i = 1,2,3)$ in (13) are used to take into account the model of the system, that is, the relation between $\dot{x}^h(t_1,t_2)$, $x^h(t_1,t_2)$ and $x^h(t_1-\tau_1(t_1),t_2)$. The key idea behind is to consider $\dot{x}(t_1,t_2)$ as a variable in the first derivative of the Lyapunov-Krasovskii functional.

Computing the derivative of $V_1(t_1,t_2)$ along the trajectories of (2) gives:

$$\frac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)}\frac{\partial x^h(t_1,t_2)}{\partial t_1} = 2x^{hT}(t_1,t_2)P_1\dot{x}^h(t_1,t_2) + x^{hT}(t_1,t_2)Q_1x^h(t_1,t_2) - (1-\dot{\tau}_1(t_1))x^{hT}(t_1-\tau_1(t_1),t_2)Q_1x^h(t_1-\tau_1(t_1),t_2) + h_1\dot{x}^{hT}(t_1,t_2)R_1\dot{x}^h(t_1,t_2) - \int_{t_1-h_1}^{t_1} \dot{x}^{hT}(s,t_2)R_1\dot{x}^h(s,t_2)ds$$

Then, for any matrices $S_0$, $S_1$, $S_2$, $N_1$, $N_2$ and $N_3$ using (12) and (13), we can bound the derivative as follows:

$$\begin{aligned}
&\frac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)}\frac{\partial x^h(t_1,t_2)}{\partial t_1} \leq \\
&2x^{hT}(t_1,t_2)P_1\dot{x}^h(t_1,t_2) + x^{hT}(t_1,t_2)Q_1x^h(t_1,t_2) \\
&-(1-d_1)x^{hT}(t_1-\tau_1(t_1),t_2)Q_1x^h(t_1-\tau_1(t_1),t_2) + h_1\dot{x}^{hT}(t_1,t_2)R_1\dot{x}^h(t_1,t_2) \\
&-\int_{t_1-\tau_1(t_1)}^{t_1}\dot{x}^{hT}(s,t_2)R_1\dot{x}^h(s,t_2)ds \\
&+2\left[x^{hT}(t_1,t_2)S_0 + x^{hT}(t_1-\tau_1(t_1),t_2)S_1 + \dot{x}^{hT}(t_1,t_2)S_2\right]\times \\
&\left\{x^h(t_1,t_2) - x^h(t_1-\tau_1(t_1),t_2) - \int_{t_1-\tau_1(t_1)}^{t_1}\dot{x}^h(s,t_2)ds\right\} \\
&+2\left[x^{hT}(t_1,t_2)N_1 + x^{hT}(t_1-\tau_1(t_1),t_2)N_2 + \dot{x}^{hT}(t_1,t_2)N_3\right]\times \\
&\left\{\dot{x}^h(t_1,t_2) - A_{11}x^h(t_1,t_2) - A_{12}x^v(t_1,t_2) - A_{11d}x^h(t_1-\tau_1(t_1),t_2) - A_{12d}x^v(t_1,t_2-\tau_2(t_2))\right\}
\end{aligned}$$

Further, using the newly defined vector $\xi(t_1,t_2)$, the expression above can be rewritten as

$$\begin{aligned}
&\frac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)}\frac{\partial x^h(t_1,t_2)}{\partial t_1} \leq \\
&\xi^T(t_1,t_2)\left\{2e_2^TP_1e_5 + e_1^TQ_1e_1 - (1-d_1)e_3^TQ_1e_3 + h_1e5^TR_1e_5\right\}\xi(t_1,t_2) \\
&-\int_{t_1-\tau_1(t_1)}^{t_1}\zeta(t_1,t_2,s,k)^Te_7^TR_1e_7\zeta(t_1,t_2,s,k)ds \\
&+2\xi^T(t_1,t_2)\left\{e_1^TS_0e_1 - e_1^TS_0e_3 + e_3^TS_1e_1 - e_3^TS_1e_3 + e_5^TS_2e_1 - e_5^TS_2e_3\right\}\xi(t_1,t_2) \\
&-2\int_{t_1-\tau_1(t_1)}^{t_1}\zeta(t_1,t_2,s,k)^T\left\{e_1^TS_0e_7 + e_3^TS_1e_7 + e_5^TS_2e_7\right\}\zeta(t_1,t_2,s,k)ds \\
&+2\xi^T(t_1,t_2)\left\{e_1^TN_1e_5 - e_1^TN_1A_{11}e_1 - e_1^TN_1A_{12}e_2 - e_1^TN_1A_{11d}e_3 - e_1^TN_1A_{12d}e_4\right. \\
&e_3^TN_2e_5 - e_3^TN_2A_{11}e_1 - e_3^TN_2A_{12}e_2 - e_3^TN_2A_{11d}e_3 - e_3^TN_2A_{12d}e_4 \\
&\left.e_5^TN_3e_5 - e_5^TN_3A_{11}e_1 - e_5^TN_3A_{12}e_2 - e_5^TN_3A_{11d}e_3 - e_5^TN_3A_{12d}e_4\right\}\xi(t_1,t_2).
\end{aligned}$$

We get similar expression for the second direction, which combined with the inequality obtained above yield the following condition for both directions

$$
\begin{aligned}
\nabla_{\varsigma} V(x(t_1,t_2)) \leq{}& \xi^T(t_1,t_2)\left\{2e_2^T P_1 e_5 + e_1^T Q_1 e_1 - (1-d_1)e_3^T Q_1 e_3 + h_1 e5^T R_1 e_5\right\}\xi(t_1,t_2)\\
& - \int_{t_1-\tau_1(t_1)}^{t_1} \zeta(t_1,t_2,s,k)^T e_7^T R_1 e_7 \zeta(t_1,t_2,s,k)ds\\
& + 2\xi^T(t_1,t_2)\left\{e_1^T S_0 e_1 - e_1^T S_0 e_3 + e_3^T S_1 e_1 - e_3^T S_1 e_3 + e_5^T S_2 e_1 - e_5^T S_2 e_3\right\}\xi(t_1,t_2)\\
& - 2\int_{t_1-\tau_1(t_1)}^{t_1} \zeta(t_1,t_2,s,k)^T\left\{e_1^T S_0 e_7 + e_3^T S_1 e_7 + e_5^T S_2 e_7\right\}\zeta(t_1,t_2,s,k)ds\\
& + 2\xi^T(t_1,t_2)\left\{e_1^T N_1 e_5 - e_1^T N_1 A_{11} e_1 - e_1^T N_1 A_{12} e_2 - e_1^T N_1 A_{11d} e_3 - e_1^T N_1 A_{12d} e_4\right.\\
& e_3^T N_2 e_5 - e_3^T N_2 A_{11} e_1 - e_3^T N_2 A_{12} e_2 - e_3^T N_2 A_{11d} e_3 - e_3^T N_2 A_{12d} e_4\\
& \left. e_5^T N_3 e_5 - e_5^T N_3 A_{11} e_1 - e_5^T N_3 A_{12} e_2 - e_5^T N_3 A_{11d} e_3 - e_5^T N_3 A_{12d} e_4\right\}\xi(t_1,t_2)\\
& + \xi^T(t_1,t_2)\left\{e_2^T P_2 e_6 + e_2^T Q_2 e_2 - (1-d_2)e_3^T Q_1 e_3 + h_2 e_6^T R_2 e_6\right\}\xi(t_1,t_2)\\
& - \int_{t_2-\tau_2(t_2)}^{t_2} \zeta(t_1,t_2,s,k)^T e_8^T R_2 e_8 \zeta(t_1,t_2,s,k)ds\\
& + 2\xi^T(t_1,t_2)\left\{e_2^T T_0 e_2 - e_2^T T_0 e_4 + e_4^T T_1 e_2 - e_4^T T_1 e_4 + e_6^T T_2 e_2 - e_6^T T_2 e_4\right\}\xi(t_1,t_2)\\
& - 2\int_{t_2-\tau_2(t_2)}^{t_2} \zeta(t_1,t_2,s,k)^T\left\{e_2^T T_0 e_8 + e_4^T T_1 e_8 + e_6^T T_2 e_8\right\}\zeta(t_1,t_2,s,k)ds\\
& 2\xi^T(t_1,t_2)\left\{e_2^T M_1 e_6 - e_2^T M_1 A_{21} e_1 - e_2^T M_1 A_{21d} e_3 - e_2^T M_1 A_{22d} e_4 - e_2^T M_1 A_{22d} e_4\right.\\
& e_4^T M_2 e_6 - e_4^T M_2 A_{21} e_1 - e_4^T M_2 A_{22} e_2 - e_4^T M_2 A_{21d} e_3 - e_4^T M_2 A_{22d} e_4\\
& \left. e_6^T M_3 e_6 - e_6^T M_3 A_{21} e_1 - e_6^T M_3 A_{22} e_2 - e_6^T M_3 A_{21d} e_3 - e_6^T M_3 A_{22d} e_4\right\}\xi(t_1,t_2).
\end{aligned}
$$

Let

$$
\tilde{H} = diag\{H \quad H \quad H\}, \quad X^v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ * & X_{11}^v & 0 & X_{12}^v & 0 & X_{13}^v\\ * & * & 0 & 0 & 0 & 0\\ * & * & * & X_{22}^v & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & * & X_{33}^v \end{bmatrix} \geq 0
$$

Then, applying lemma 1, respectively for $X^h$ and $X_v$, we get

$$\tilde{H}\xi(t_1,t_2)X\xi(t_1,t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \xi^T(t_1,t_2)X^h\xi(t_1,t_2)ds - \int_{t_1,t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1,t_2)X^v\xi(t_1,t_2)dk$$
$$= \tilde{H}\xi(t_1,t_2)X\xi(t_1,t_2) -$$
$$\int_{t_1-\tau_1(t_1)}^{t_1}\int_{t_1,t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1,t_2)\left\{diag\left\{\frac{1}{\tau_2(t_2)} \quad 0 \quad \frac{1}{\tau_2(t_2)} \quad 0 \quad \frac{1}{\tau_2(t_2)} \quad 0\right\}X^h\right.$$
$$\left.+diag\left\{0 \quad \frac{1}{\tau_1(t_1)} \quad 0 \quad \frac{1}{\tau_1(t_1)} \quad 0 \quad \frac{1}{\tau_1(t_1)}\right\}X^v\right\}\xi(t_1,t_2)dsdk$$
$$\leq \tilde{H}\xi(t_1,t_2)X\xi(t_1,t_2) -$$
$$\int_{t_1-\tau_1(t_1)}^{t_1}\int_{t_1,t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1,t_2)\left\{diag\left\{\frac{1}{h_2} \quad 0 \quad \frac{1}{h_2} \quad 0 \quad \frac{1}{h_2} \quad 0\right\}X^h\right.$$
$$\left.+diag\left\{0 \quad \frac{1}{h_1} \quad 0 \quad \frac{1}{h_1} \quad 0 \quad \frac{1}{h_1}\right\}X^v\right\}\xi(t_1,t_2)dsdk$$

$$\tilde{H}\xi^T(t_1,t_2)X\xi(t_1,t_2) - \int_{t_1-\tau_1(t_1)}^{t_1}\int_{t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1,t_2)\tilde{U}X\xi(t_1,t_2)ds \qquad (14)$$

with

$$\tilde{U} = diag\{U \quad U \quad U\}$$

$$\nabla_\varsigma V(x(t_1,t_2)) \leq \xi^T(t_1,t_2)\Xi\xi(t_1,t_2) - \int_{t_1-\tau_1(t_1)}^{t_1}\int_{t_2-\tau_2(t_2)}^{t_2} \zeta^T(t_1,t_2,s,k)\Psi\zeta(t_1,t_2,s,k)dsdk$$

with

$$\Xi = \begin{bmatrix} Q - \mathrm{sym}\{\Lambda_1 A\} + \mathrm{sym}\{\Upsilon_0\} + HX_{11} & A^T\Lambda_2^T - \Lambda_1 A_d + \Upsilon_{10} + HX_{12} & P + \Lambda_1 - A^T\Lambda_3^T + \Upsilon_2^T + HX_{13} \\ * & -(I_n - W)Q - \mathrm{sym}\{\Lambda_2 A_d\} - \mathrm{sym}\{\Upsilon_1\} + HX_{22} & \Lambda_2 - A_d^T\Lambda_3^T - \Upsilon_2^T + HX_{23} \\ * & * & HR + \mathrm{sym}\{\Lambda_3\} + HX_{33} \end{bmatrix} < 0$$

and

$$\Psi = \begin{bmatrix} UX_{11} & UX_{12} & UX_{13} & U\Upsilon_0 \\ * & UX_{22} & UX_{23} & U\Upsilon_1 \\ * & * & UX_{33} & U\Upsilon_2 \\ * & * & * & UR \end{bmatrix}$$

If $\Xi \prec 0$ and $\Psi \geq 0$, then $\nabla_\varsigma V(x(t_1,t_2)) \prec -\epsilon\|x(t_1,t_2)\|^2$ for a sufficiently small $\epsilon$, which ensures the asymptotic stability of system (2).

Specifically, if we select $R \succ 0$ then $X$ can be chosen to be $X = \begin{bmatrix}\Upsilon_0 \\ \Upsilon_1 \\ \Upsilon_2\end{bmatrix} R^{-1}\left[\Upsilon_0^T \ \Upsilon_1^T \ \Upsilon_2^T\right] \geq 0$. This ensures that $\Psi \geq 0$. In this case, $\Xi \prec 0$ is equivalent to $\Phi \prec 0$ according to the Schur complement.

### 6.2 Proof of Theorem 2

Using the same Lyapunov functional as mentioned in section 6.1, for the closed loop system, we get the condition

$$\Phi_c = \begin{bmatrix} Q - \text{sym}\{\Lambda_1 A_c\} + \text{sym}\{\Upsilon_0\} & A_c^T \Lambda_2^T - \Lambda_1 A_d + \Upsilon_{10} & P + \Lambda_1 - A_c^T \Lambda_3^T + \Upsilon_2^T & HU\Upsilon_0 \\ * & -(I_n - W)Q - \text{sym}\{\Lambda_2 A_d\} - \text{sym}\{\Upsilon_1\} & \Lambda_2 - A_d^T \Lambda_3^T - \Upsilon_2^T & HU\Upsilon_1 \\ * & * & HR + \text{sym}\{\Lambda_3\} & HU\Upsilon_2 \\ * & * & * & -HUR \end{bmatrix} < 0,$$

that is, according to Theorem 1, is sufficient to ensure that the closed loop system is stable.

Consider the case where $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda$, then

$$\Phi_c = \begin{bmatrix} Q - \text{sym}\{\Lambda A_c\} + \text{sym}\{\Upsilon_0\} & A_c^T \Lambda^T - \Lambda A_d + \Upsilon_{10} & P + \Lambda - A_c^T \Lambda^T + \Upsilon_2^T & HU\Upsilon_0 \\ * & -(I_n - W)Q - \text{sym}\{\Lambda A_d\} - \text{sym}\{\Upsilon_1\} & \Lambda - A_d^T \Lambda^T - \Upsilon_2^T & HU\Upsilon_1 \\ * & * & HR + \text{sym}\{\Lambda\} & HU\Upsilon_2 \\ * & * & * & -HUR \end{bmatrix} < 0 \quad (15)$$

Note that this last condition is bilinear with respect to the variables $\Lambda$ and $K$ and therefore it may be considered as a BMI problem. To obtain LMI (7), it is necessary to pre- and post-multiply inequality (15) by $diag\left\{\Lambda^{-1}, \Lambda^{-1}, \Lambda^{-1}, \Lambda^{-1}\right\}$.

## Références


1. M. Benhayoun, A. Benzaouia, F. Mesquine, and F. Tadeo. Stabilization of 2D continuous systems with multi-delays and saturated control. In *18th Mediterranean Conference on Control and Automation, MED'10 - Conference Proceedings*, pages 993–999, 2010.
2. J. Bochniak and K. Galkowski. LMI-based analysis for continuous-discrete linear shift invariant nD-systems. *Journal of Circuits, Systems and Computers*, 14(2):307–332, 2005.
3. L. Boyd, L.El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.
4. S. Chen. Stability analysis for 2D systems with interval time-varying delays and saturation nonlinearities. *Signal Processing*, 90(7):2265–2275, 2010.
5. S.F. Chen and I.K. Fong. Robust filtering for 2D state-delayed systems with NFT uncertainties. *Signal Processing, IEEE Transactions on*, 54(1):274–285, 2006.
6. M. Ghamgui, N. Yeganefar, O. Bachelier, and D. Mehdi. Asymptotic stability of 2D continuous time varying delay systems. *13th International conference on Sciences and Techniques of Automatic control & computer engineering*, 2011.

7. M. Ghamgui, N. Yeganefar, O. Bachelier, and D. Mehdi. Stability and stabiliszation of continuous state delayed systems. *IEEE Conference on Decision and Control*, 2011.
8. Y. He, M. Wu, J.H. She, and G.P. Liu. Parameter-dependent lyapunov functional for stability of time-delay systems with polytopic-type uncertainties. *Automatic Control, IEEE Transactions on*, 49(5):828–832, 2004.
9. A. Hmamed, F. Mesquine, F. Tadeo, M. Benhayoun, and A. Benzaouia. Stabilization of 2D saturated systems by state feedback control. *Multidimensional Systems and Signal Processing*, pages 277–292, 2010.
10. T. Kaczorek. LMI approach to stability of 2D positive systems. *Multidimensional Systems and Signal Processing*, 20(1):39–54, 2009.
11. W. Paszke, J. Lam, K. Galkowski, S. Xu, and Z. Lin. Robust stability and stabilization of 2D discrete state-delayed systems. *Systems and Control Letters*, 51:277–291, 2004.
12. R.P. Roesser. Discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control*, 20(1):1–10, 1975.
13. L. Wang, S. Mo, D. Zhou, and F. Gao. Robust design of feedback integrated with iterative learning control for batch processes with uncertainties and interval time-varying delays. *Journal of Process Control*, 2011.
14. M. Wu, Y. He, J. She, and G. Liu. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40(8):1435–1439, 2004.
15. J.M. Xu and L. Yu. $H_\infty$ control for 2D discrete state delayed systems in the second FM model. *Zidonghua Xuebao/Acta Automatica Sinica*, 34(7):809–813, 2008.
16. S. Ye, W. Wang, and Y. Zou. Robust guaranteed cost control for a class of two-dimensional discrete systems with shift-delays. *Multidimensional Systems and Signal Processing*, 20(3):297–307, 2009.
17. Z. Zuo and Y. Wang. Robust stability and stabilisation for nonlinear uncertain time-delay systems via fuzzy control approach. *IET Control Theory and Applications*, 1(1):422–429, 2007.